1. Let \( w = f(x, y) \) where \( x = u + v \) and \( y = uv \).

   a) Find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) in terms of partial derivatives of \( w \) with respect to \( x \) and \( y \).

   b) Find \( \frac{\partial^2 w}{\partial u \partial v} \).

   a) First use the chain rule for 1 variable namely

   \[
   \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} v \quad (1)
   \]

   and also

   \[
   \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} u \quad (2)
   \]

   b) For this part what we need to do is realize that \( \frac{\partial^2 w}{\partial u \partial v} = \frac{\partial w}{\partial u} \frac{\partial w}{\partial v} \) thus we must differentiate each object carefully. Start with \( \frac{\partial w}{\partial u} \). We know that really this is \( h(u, v) = \frac{\partial w}{\partial x} (x, y) \). Now we must take \( \frac{\partial h}{\partial u} \) and use the chain rule for this. Using the chain rule this equals

   \[
   \frac{\partial h}{\partial u} = \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial^2 w}{\partial u \partial x} + \frac{\partial^2 w}{\partial y \partial x} u \quad (3)
   \]

   Then evaluating the first partials we get that this equals

   \[
   \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} v \quad (4)
   \]

   Now if we take the right hand side of (2) if we want to take \( \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial y} u \right) \) then we must use both the product rule and the chain rule.

   Using the product rule we get that

   \[
   \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial y} u \right) = \left( \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial y} \right) \right) u + \frac{\partial w}{\partial y} \quad (5)
   \]

   Now to get the last ingredient in our recipe for chain rule fun we must use the
chain rule again, This time on the function $g(u, v) = \frac{\partial w}{\partial y}(x, y)$. This gives us
the following

$$\frac{\partial g}{\partial u} = \frac{\partial}{\partial u} \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} \frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y^2} \frac{\partial y}{\partial u}$$

(6)

then adding in the first partials we get that this equals

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} v$$

(7)

Taking all of the ingredients together and mixing gives us that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} u + \frac{\partial^2 w}{\partial y^2} uv + \frac{\partial w}{\partial y}$$

(8)

If conditions on $f$ are nice enough we have that $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$ and thus that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} (u + v) + \frac{\partial^2 w}{\partial y^2} uv + \frac{\partial w}{\partial y}$$

(9)

Now let us do some double integrals

Problem 26 asks us to calculate the mass and the center of mass for the
lamina bounded by $x = y^2$ and $x = 4$ with the given density $\rho(x, y) = y + 3$.

We are in the following region enclosed in red
Let us take $M$ the center of mass given on this region by

$$M = \int \int_R \rho(x,y)dA = \int_0^4 \int_{\sqrt{x}}^{-\sqrt{x}} (y + 3)\,dxdy = \int_0^4 \frac{y^2}{2} + 3y\sqrt{\pi}$$  \quad (10)$$

Taking (10) we get that this equals

$$M = \int_0^4 \left(\frac{x}{2} + 3\sqrt{x}\right) - \left(\frac{x}{2} - 3\sqrt{x}\right)\,dx = \int_0^4 6\sqrt{x}\,dx = 4x^{\frac{3}{2}}\bigg|_0^4 = 32.$$  \quad (11)

Then we have that for $M_x$ we have that

$$M_x = \int \int_R y\rho(x,y)dA = \int_0^4 \int_{\sqrt{x}}^{-\sqrt{x}} (y^2 + 3y)\,dydx = \int_0^4 \left(\frac{y^3}{3} + \frac{3y^2}{2}\sqrt{\pi}\right)\,dx$$  \quad (12)

reducing (12) we find that

$$M_x = \int_0^4 \frac{2}{3}x^{\frac{5}{2}}\,dx = \frac{4}{15}x^{\frac{5}{2}}\bigg|_0^4 = \frac{128}{15}.$$  \quad (13)
For $M_y$ we then similarly compute that

$$M_y = \int \int_R x \rho(x, y) \, dA = \int_0^4 \int_{\sqrt{x}}^{\sqrt{x}} (xy + 3x) \, dy \, dx = \int_0^4 x \frac{y^2}{2} + 3xy|_{\sqrt{x}}^{\sqrt{x}} \, dx$$

expanding (14) we find that it equals

$$M_y = \int_0^4 \left( (x^2 + 3x^2) - (x^2 - 3x^2) \right) \, dx = \int_0^4 6x^2 \, dx = \frac{12}{5} x^2 \bigg|_0^4 = \frac{384}{5}$$

Putting (11), (13) and (15) we have that the center of mass is $\left( \frac{M_y}{M} , \frac{M_x}{M} \right) = \left( \frac{12}{5}, \frac{4}{5} \right)$.

Hopefully we all understand lamina and double integrals better now.