In exercises 1,3, and 5 find the equation of the tangent plane and the normal line to the surface at the given point.

1. \( f(x, y) = z = x^2 + y^2 - 1 \) at \((2, 1, 4)\) and \((2, 0, 3)\).

We know that \( f_x = 2x \) and \( f_y = 2y \) so the tangent plane is

\[
z - 4 = 4(x - 2) + 2(y - 1) \quad (1)
\]

and the normal line is

\[
x = 2 + 4t \quad (2)
\]
\[
y = 1 + 2t \quad (3)
\]
\[
z = 4 - t \quad (4)
\]

since we take the slope vector \((4, 2, -1)\) and make a line that goes through \((2, 1, 4)\).

For the point \((2, 0, 3)\) the tangent plane is

\[
z - 3 = 4(x - 2) \quad (5)
\]

and the normal line is

\[
x = 2 + 4t \quad (6)
\]
\[
y = 0 \quad (7)
\]
\[
z = 3 - t \quad (8)
\]

3. \( f(x, y) = z = \sin(x) \cos(y) \) at \((0, \pi, 0)\) and \((\frac{\pi}{2}, \pi, -1)\).

We know that \( f_x = \cos(x) \cos(y) \) and \( f_y = -\sin(x) \sin(y) \) so the tangent
plane is

\[ z = -(x - \pi) \quad (9) \]

and the normal line is

\[ x = -t \quad (10) \]
\[ y = \pi \quad (11) \]
\[ z = -t \quad (12) \]

since we take the slope vector \((-1, 0, -1)\) and make a line that goes through \((0, \pi, 0)\).

For the point \((\pi/2, \pi, -1)\) the tangent plane is

\[ z = -1 \quad (13) \]

and the normal line is

\[ x = \frac{\pi}{2} \quad (14) \]
\[ y = \pi \quad (15) \]
\[ z = -t \quad (16) \]

5. \( f(x, y) = z = \sqrt{x^2 + y^2} \) at \((-3, 4, 5)\) and \((8, -6, 10)\).

We know that

\[ f_x = \frac{1}{2} 2x(x^2 + y^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2 + y^2}} \quad (17) \]

and

\[ f_y = \frac{1}{2} 2y(x^2 + y^2)^{-\frac{1}{2}} = \frac{y}{\sqrt{x^2 + y^2}} \quad (18) \]
so the tangent plane is

\[ 5(z - 5) = -3(x + 3) + 4(y - 4) \] (19)

and the normal line is

\[
\begin{align*}
x &= -3 - \frac{3}{5}t \\
y &= 4 + \frac{4}{5}t \\
z &= 5 - t
\end{align*}
\] (20, 21, 22)

since we take the slope vector \((-\frac{3}{5}, \frac{4}{5}, -1)\) and make a line that goes through \((-3, 4, 5)\).

For the point \((8, -6, 10)\) the tangent plane is

\[ 5(z - 10) = 4(x - 8) - 3(y + 6) \] (23)

and the normal line is

\[
\begin{align*}
x &= 8 + \frac{4}{5}t \\
y &= -6 - \frac{3}{5}t \\
z &= 10 - t
\end{align*}
\] (24, 25, 26)

In exercises 7, 9, 11 compute the linear approximation of the function at the point.

7. \( f(x, y) = \sqrt{x^2 + y^2} \) at \((3, 0)\) and \((0, -3)\). We know from the previous part of the problem that

\[ f_x = \frac{1}{2}2x(x^2 + y^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2 + y^2}} \] (27)
and

\[ f_y = \frac{1}{2} 2y(x^2 + y^2)^{-\frac{3}{2}} = \frac{y}{\sqrt{x^2 + y^2}} \]  

so

\[ L(x, y) = 3 + (x - 3) \]  

at (3,0). And at (0,−3) we know that

\[ L(x, y) = 3 + (y - 3) \]

9. \( f(x, y) = xe^{xy^2} + 3y^2 \) at (0,1) and (2,0). We know that

\[ f_x = e^{xy^2} + xy^2e^{xy^2} \]  

and

\[ f_y = 2yxe^{xy^2} + 6y \]  

so

\[ L(x, y) = 3 + x + 6(y - 1) \]  

at (0,1). And at (2,0) we know that

\[ L(x, y) = 2 + (x - 2) \]

11. \( f(w, x, y, z) = w^2xy - e^{wyz} \) at (−2,3,1,0) and (0,1,−1,2). We know that

\[ f_w = 2wxy - yze^{wyz} \]  

\[ f_x = w^2y \]  

\[ f_y = w^2x - wze^{wyz} \]
\[ f_z = -wy e^{wy} \]  

so

\[ L(w, x, y, z) = 12 - 12(w + 2) + 4(x - 3) + 12(y - 1) + 6z \]  

at \((-2, 3, 1, 0)\). And at \((0, 1, -1, 2)\) we know that

\[ L(x, y) = -1 + 2w \]

In the exercises 23 and 25 find the increment \(\Delta z\) and write it in the form given in definition 4.1.

23. \(f(x, y) = z = 2xy + y^2\)

\[ \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = 2(x + \Delta x)(y + \Delta y) + (y + \Delta y)^2 - 2xy - y^2 \]  

After expanding (41) we find that

\[ \Delta z = (2x + 2y)\Delta y + 2y\Delta x + (2\Delta x + 2\Delta y)(\Delta y) = f_y \Delta y + f_x \Delta x + \epsilon_1 \Delta y \]  

where \(\epsilon_1 = 2\Delta x + 2\Delta y\)

25. \(f(x, y) = z = x^2 + y^2\)

\[ \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = (x + \Delta x)^2 + (y + \Delta y)^2 - x^2 - y^2 \]  

Then expanding (43) we arrive at

\[ \Delta z = 2x \Delta x + 2y \Delta y + (\Delta x)^2 + (\Delta y)^2 \]
Then here if we let $\epsilon_1 = \Delta x$ and $\epsilon_2 = \Delta y$ then (44) equals

$$\Delta z = f_x \Delta x + f_y \Delta y + \epsilon_1 (\Delta x) + \epsilon_2 (\Delta y)$$ (45)

27. Determine whether or not $f(x, y) = z = x^2 + 3xy$ is differentiable.

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y) = (x+\Delta x)^2 + 3(x+\Delta x)(y+\Delta y) - x^2 - 3xy$$ (46)

After expanding eqref27.1 we arrive at

$$\Delta z = (2x+3y)\Delta x + 3x\Delta y + (\Delta y + \Delta x)\Delta x$$ (47)

We know that after labeling $\epsilon_1 = \Delta y + \Delta x$ we know that

$$\Delta z = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x$$ (48)

Hence it is differentiable.

In exercise 29 find the total differential of $f(x, y)$

29. $f(x, y) = z = ye^x + \sin(x)$. We know that $f_x = ye^x + \cos(x)$ and that $f_y = e^x$ so we know that $dz = (ye^x + \cos(x))dx + e^xdy$.

In exercise 31 show that the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but the function $f(x, y)$ is not differentiable at $(0, 0)$.

31. Let

$$f(x, y) = \begin{cases} 2xy/(x^2+y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$ (49)

observe that

$$f_x = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$ (50)
and
\[ f_y = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = 0 \] (51)

however if we try to take \( \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} \) at \( y = x \) then this equals \( \lim_{y \to 0} \frac{2y^2}{2y^2} = 1 \). Thus the function \( f(x, y) \) is not continuous at \((0,0)\), hence it is not differentiable at \((0,0)\).

43. Show that

\[ f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases} \] (52)

It is continuous since for all points \((x, y) \neq (0,0)\) the function \( f(x, y) \) is clearly continuous, and at \((0,0)\) we have that

\[ \lim_{(x,y) \to (0,0)} |f(x, y)| \leq \lim_{y \to 0} |y| = 0 \] (53)

thus it is continuous. However, at \( f_x(0,0) = f_y(0,0) = 0 \). This means that if \( f(x, y) \) is differentiable that \( f(x, y) = 0 + \epsilon_1 x + \epsilon_2 y \), but along \( x = y \) we know that \( f(x, x) = \frac{1}{2} x = 0 + \epsilon_1 x + \epsilon_2 x \) where \( \epsilon_1(x, x) \) goes to 0 as \( x \to 0 \) and similarly with \( \epsilon_2(x, x) \). This is a contradiction since \( \frac{1}{2} x \) is linear while \( \epsilon_1 x + \epsilon_2 x \) is not linear.