STURDY HARMONIC FUNCTIONS AND THEIR INTEGRAL REPRESENTATIONS

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Abstract. Sturdy harmonic functions constitute all but the least tractable of the positive harmonic functions in potential-theoretic settings. They are the uniform limits on compact sets of positive, bounded harmonic functions and are also produced by a simple integral representation on the boundary of a natural compactification of the space on which they are defined. The boundary of that compactification is metrizable, and more regular for the Dirichlet problem, in general, than is the Martin boundary if that boundary is even defined in the setting.

1. Introduction

In classical potential theory, R. S. Martin [22] constructed what is now called the Martin compactification of a Euclidean domain X. His aim was to obtain an integral representation of all positive harmonic functions. This representation can now be considered as an example of Choquet's integral representation of compact convex sets in a locally compact Hausdorff space. Unfortunately, as shown in [25], the boundary of Martin's compactification, i.e., the Martin boundary, may have irregularities with respect to the Dirichlet problem that form a set of positive harmonic measure. In [19] and [20], the second author of this paper constructed a new compactification $\widehat{X}$ of X that in general is better than the Martin boundary for solving the Dirichlet problem. When using $\widehat{X}$, some special positive harmonic functions may not have an integral representation. On the other hand, the boundary of $\widehat{X}$ enjoys nice regularity properties with respect to the Dirichlet problem. In addition, the construction of $\widehat{X}$ works in the framework of harmonic

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spaces; it can therefore be applied to a wide class of linear, elliptic and parabolic second order differential operators.

In this paper we shall give a new construction of the compactification $\widehat{X}$ using only harmonic functions and not potentials. Here, for a special setting, is a rough outline of what we shall do. The reader may assume that we are working in a favored potential-theoretic setting, but to be exact, we suppose that $(X, \mathcal{H})$ is a harmonic space in the sense of [11] satisfying Doob's convergence axiom with $1 \in \mathcal{H}(X)$. We also suppose that r is a normalized reference measure on X, i.e., X is the smallest absorbing set containing the support of r. For ordinary harmonic functions, one may let r be unit mass at a point. Now by a result of K. Janßen [17], the set

$$\mathcal{H}_r^1(X) = \{h \in \mathcal{H}(X) : h \geq 0, \quad \int h \, dr \leq 1\}$$

is a Choquet simplex. We let $\mathcal{E}^1$ denote the set of extreme points of $\mathcal{H}_r^1(X)$. There exists a unique measure μ_1 on $\mathcal{H}_r^1(X)$ supported by $\mathcal{E}^1$ such that for all $x \in X$,

$$1 = \int_{\mathcal{E}^1} g(x) \, \mu_1(dg).$$

We define $Q : X \times X \longrightarrow \mathbb{R}^+$ by setting

$$Q(x, y) := \int_{\mathcal{E}^1} g(x) g(y) \, \mu_1(dg) \ , \ x, y \in X.$$

The family $\mathcal{Q} := \{Q(x, \cdot) : x \in X\}$ is a set of positive, bounded harmonic functions on X. The $\mathcal{Q}$-compactification $\widehat{X}$ of X is the "smallest" compactification of X such that every function in $\mathcal{Q}$ can be continuously extended to $\widehat{X}$ with the extensions separating the boundary points; this is the compactification used in this paper. To establish the numerous properties of $\widehat{X}$ we first show that it is a quotient of the larger Feller compactification. It then turns out that functions that are exactly the uniform limits on compact sets of sequences of bounded harmonic functions allow a nice integral representation on $\widehat{X}$. We have called these functions *sturdy harmonic functions.* In developing their properties, we give several equivalent conditions that force them to have an integral representation even with respect to minimal representing measures on the boundary of $\widehat{X}$. Several examples given by the Laplace equation and the heat equation show that $\widehat{X}$ is in general different from the Martin compactification; it is, however, the same for ordinary harmonic functions on Lipschitz domains. Conditions are also presented that force all positive harmonic functions to be sturdy, extending the results first presented in [4]. Based on earlier work of the authors in [5], [6], and [7] concerning the boundary behavior of harmonic functions, the second author and H. von Weizsäcker have shown that a required condition is naturally satisfied when the underlying measure space is second countable. That result constitutes the appendix forming the last section of this paper. The result there is used earlier in our paper to obtain a boundary limit theorem for sturdy harmonic function.

2. Sturdy Harmonic Functions

In what follows, we will work on a locally compact Hausdorff space X with a countable base. We will assume that the space $\mathcal{C}(X)$ of real-valued continuous functions on X and all subspaces of $\mathcal{C}(X)$ are endowed with the topology of uniform convergence on compact subsets of X, i.e., the ucc-topology. Given any set $\mathcal{F} \subseteq \mathcal{C}(X)$, we will let $\mathcal{F}^+$ denote the set of nonnegative functions and $\mathcal{F}_b$ the set of bounded functions in $\mathcal{F}$. We will write $f \vee g$ and $f \wedge g$ respectively for the pointwise upper and lower envelopes of two functions f and g in $\mathcal{C}(X)$.

There are only a few formal requirements needed to show that an appropriate space of continuous functions has a nice integral representation on a natural compactification of X. Our formal assumptions about such a space will be given in the next section. First, we want to motivate our assumptions by showing that they hold for any reasonable potential-theoretic setting. The reader may choose a favorite example. Formally, in this section, we let $(X, \mathcal{H})$ be a harmonic space in the sense of Constantinescu–Cornea [11] such that the constant function 1 is harmonic and Doob's convergence axiom is satisfied.

A positive measure r on X is called a *reference measure* if X is the smallest absorbing set containing the support of r. In terms of the usual notation, we write $A_r = X$. Recall that for classical harmonic functions or a Brelot harmonic space [9], r may be a unit mass at some point of the connected domain. For more general settings, r may be an atomic probability measure with the atoms at the points of a countable dense set in the domain.

It now follows from a result of Janßen [17] that the set

$$\mathcal{H}_r^1(X) := \left\{ h \in \mathcal{H}^+(X) : \int h \, dr \leq 1 \right\}$$

is a Choquet simplex. In particular, it is a compact subset of $\mathcal{H}^+$ with respect to the ucc-topology.

2.1. **Definition.** *A positive harmonic function $h \in \mathcal{H}^+(X)$ is called **sturdy** if there exists a sequence $\langle h_n \rangle \subset \mathcal{H}_b^+(X)$ converging to h in the ucc-topology. If, moreover, there exists a finite reference measure r on X such that the functions h_n are all contained in $\mathcal{H}_r^1(X)$, then h is called r-**sturdy**.*

Clearly, for ordinary harmonic functions or a Brelot harmonic space, a sturdy harmonic function is r-sturdy with respect to a unit point mass r. To connect the two notions in the general case, we need the following result.

2.2. **Lemma.** *Let $h : X \longrightarrow \mathbb{R}^+$, and let $\langle x_n \rangle$ be a sequence in X. Let $\langle h_n \rangle$ be a sequence of bounded, positive functions on X such that for all $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} h_n(x_m) = h(x_m).$$

Then there exists a subsequence $\langle g_n \rangle$ of $\langle h_n \rangle$ and a sequence of positive numbers α_n such that for all $k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \alpha_n \leq 1, \quad \sum_{n=1}^{\infty} \alpha_n h(x_n) \leq 1, \ \textit{and} \ \sum_{n=1}^{\infty} \alpha_n g_k(x_n) \leq 1.$$

Proof. Choose a subsequence $\langle g_n \rangle$ of $\langle h_n \rangle$ so that for each $n \in \mathbb{N}$ and each $m \leq n$,

$$|g_n(x_m) - h(x_m)| \leq 1.$$

For each $n \in \mathbb{N}$, we set

$$M_n := h(x_n) \vee \left[\max_{1\leq m\leq n} \sup_{y\in X} g_m(y) \right] \quad \text{and} \quad \alpha_n := \frac{1}{(1+M_n)\cdot 2^n}.$$

To see that $\sum_{n=1}^{\infty} \alpha_n g_k(x_n) \leq 1$ for each $k \in \mathbb{N}$, we note that for $1 \leq m \leq k$,

$$g_k(x_m) \leq h(x_m) + 1 \leq M_m + 1$$

and for $m > k$

$$g_k(x_m) \leq \sup_{y\in X} g_k(y) \leq M_m.$$

The rest is clear. □

2.3. **Proposition.** *Fix $h \in \mathcal{H}^+(X)$. Then the following conditions are equivalent:*

1. *h is sturdy.*
2. *There exists a sequence $\langle h_n \rangle \subset \mathcal{H}_b^+(X)$ converging pointwise to h.*
3. *There exists a finite reference measure r on X such that h is r-sturdy.*

Proof. By definition, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). To show that (2) $\Rightarrow$ (3), we choose a dense sequence $\langle x_n \rangle$ in X and apply Lemma 2.2 to the sequence $\langle h_n \rangle$, thus obtaining sequences $\langle g_n \rangle$ and $\langle \alpha_n \rangle$. Letting ε_{x_n} denote unit mass at x_n, we see that

$$r := \sum_{n=1}^{\infty} \alpha_n \varepsilon_{x_n}$$

is a finite reference measure on X such that $\langle g_n \rangle \subset \mathcal{H}_r^1(X)$. Since $\mathcal{H}_r^1(X)$ is compact, there exists a convergent subsequence $\langle g_{n_k} \rangle$ of $\langle g_n \rangle$ with the convergence being uniform on compact sets to h. $\square$

2.4**. Example.**
(1) Every positive, bounded harmonic function is clearly sturdy.
(2) Recall that a function $h \in \mathcal{H}^+(X)$ is called *quasi-bounded* if it is the limit of an increasing sequence $\langle h_n \rangle \subset \mathcal{H}_b^+(X)$. By Dini's theorem, all such functions are sturdy.
(3) In classical potential theory on $\mathbb{R}^n$ every positive harmonic function on $B_1(0) := \{x \in \mathbb{R}^n : \|x\| < 1\}$ is sturdy, whereas the harmonic function

$$x \longmapsto -\log |x|$$

on $B_1(0) \setminus \{0\}$ in $\mathbb{R}^2$ is not sturdy. This follows by a simple calculation. In Section 8, these assertions will be corollaries of more general results.

In what follows, we shall show that sturdy harmonic functions on X have a nice integral representation with respect to a very regular kernel and the boundary of a natural compactification of X.

3. Harmonic Systems

Again, we let X be a locally compact Hausdorff space with a countable base for the topology. Although we have potential-theoretic settings in mind, we actually need to work only with functions defined on all of X having a few special properties. Here is what we need for our results.

3.1**. Definition.** *A* ***harmonic system*** *is a triple* $(X, \mathcal{H}, r)$ *such that the following conditions are satisfied:*

(H_1) $\mathcal{H}$ *is a linear subspace of* $\mathcal{C}(X)$ *such that*
- (a) $1 \in \mathcal{H}$,
- (b) $\mathcal{H}^+ - \mathcal{H}^+$ *is a vector lattice, and*
- (c) *For any* $h \in \mathcal{H}$*, the set* $\{h \leq 0\}$ *is either empty or not compact in* X*.*

(H_2) $r : \mathcal{H}^+ \longrightarrow [0, +\infty]$ *is a lower semicontinuous function that is continuous on the set* $\{h \in \mathcal{H}^+ : h \leq 1\}$, *and* r *has the additional properties that for all* $h, g \in \mathcal{H}^+$

(a) $r(h + g) = r(h) + r(g)$,

(b) $r(\alpha h) = \alpha r(h)$ *for all* $\alpha \geq 0$

(c) $r(1) = 1$.

(H_3) *The set* $\mathcal{H}^1 := \{h \in \mathcal{H}^+ : r(h) \leq 1\}$ *is compact in the ucc-topology.*

We note that since $-1 \in \mathcal{H}$, it follows from Condition (H_1) that X is not compact. It follows from the Condition (H_2) and (H_3) that $\mathcal{H}^1$ is a cap in the convex cone $\mathcal{H}^+$. That is, $\mathcal{H}^1$ is compact and convex, and $\mathcal{H}^+ \setminus \mathcal{H}^1$ is convex. Since $\mathcal{H}^+ - \mathcal{H}^+$ is a vector lattice and X has a countable base, the compact, convex set $\mathcal{H}^1$ is a metrizable Choquet simplex. Let $\mathcal{E}$ denote the set of extreme points of $\mathcal{H}^1$. Then $\mathcal{E} = \mathcal{E}^1 \cup \{0\}$ where

$$\mathcal{E}^1 := \left\{h \in \mathcal{H}^+ : r(h) = 1 \text{ and } h \text{ lies on an extreme ray of } \mathcal{H}^+\right\}.$$

Therefore, the following abstract integral representation holds for each element $h \in \mathcal{H}^1$: There is a unique probability measure μ_h defined on $\mathcal{H}^1$ with $\mu_h(\mathcal{H}^1 \setminus \mathcal{E}) = 0$ such that for all $x \in X$,

$$h(x) = \int_{\mathcal{E}^1} g(x)\, \mu_h(dg).$$

The measure μ_h is often called the *minimal representing measure* for h.

3.2. **Example.** Let $\mathcal{H}(X)$ be the set of functions in a harmonic space satisfying Doob's convergence axiom. If $1 \in \mathcal{H}(X)$ and r is a reference measure on X with $r(X) = 1$, then

$$(X, \mathcal{H}(X), r)$$

is a harmonic system. The lower semicontinuity of r follows from Fatou's Lemma, and the continuity on the family $\{h \in \mathcal{H}^+ : h \leq 1\}$ follows from the inner regularity of r.

3.3. **Example.** Let $(X, \mathcal{H}, r)$ be a harmonic system, and fix $h_0 \in \mathcal{H}^+$ with $r(h_0) < +\infty$ such that $h_0(x) > 0$ for each $x \in X$. Let

$$\widetilde{\mathcal{H}} = \{h/h_0 : h \in \mathcal{H}\}\ .$$

Let $\widetilde{r}$ be the nonnegative, extended-real-valued function defined on $\widetilde{\mathcal{H}}^+$ by setting

$$\widetilde{r}(h/h_0) = r(h)/r(h_0),\ h \in \mathcal{H}^+\ .$$

Then $(X, \widetilde{\mathcal{H}}, \widetilde{r})$ is a harmonic system.

We conclude this section with several closure properties of harmonic systems.

3.4. **Proposition.** *If $h \in \mathcal{H}^+$ satisfies the equation $r(h) = 0$, then $h = 0$.*

Proof. For each $n \in \mathbb{N}$, $r(nh) = nr(h) = 0$. Therefore, the increasing sequence $\langle nh \rangle_{n\in\mathbb{N}} \subset \mathcal{H}^1$, and so a subsequence converges to a function $g \in \mathcal{H}^1$. Since g is real-valued, $h \equiv 0$. ☐

3.5. **Proposition.** *Let $(X, \mathcal{H}, r)$ be a harmonic system. If $\{h_i : i \in I\}$ is a family in $\mathcal{H}^+$ directed by increasing order with $\sup_{i\in I} r(h_i) < +\infty$, then the pointwise supremum $\sup_{i\in I} h_i \in \mathcal{H}^+$.*

Proof. By rescaling, we may assume that $\sup_{i\in I} r(h_i) = 1$. If for i, $j \in I$ we have $h_i \lneqq h_j$, then by Proposition 3.4 we must have $r(h_i) < r(h_j)$. We may therefore assume that $r(h_i) < 1$ for all $i \in I$ since otherwise the result is trivial. For each natural number $n \in \mathbb{N}$, choose $i_n \in I$ so that $r(h_{i_n}) > 1 - 1/n$ and the sequence $\langle h_{i_n} \rangle$ is increasing. Since $\mathcal{H}^1$ is compact, we may assume that the sequence $\langle h_{i_n} \rangle$ converges to a function $g \in \mathcal{H}^1$. Since r is lower semicontinuous but $g \geq h_{i_n}$ for each n, we have $r(g) = 1$. Fix any $j \in I$. There is an increasing sequence $\langle k_{i_n} \rangle$ in the original family such that for each n, $k_{i_n} \geq h_{i_n} \vee h_j$. A subsequence of this sequence converges to a function $g' \in \mathcal{H}^1$, but again, $r(g') = 1$. Since $g' \geq g$, it follows from Proposition 3.4 that $g' = g$, whence $h_j \leq g$. Therefore, $g = \sup_{i\in I} h_i$. ☐

3.6. **Proposition.** *The topology of pointwise convergence and the ucc-topology are the same on $\mathcal{H}^1$. Moreover, if a sequence $\langle h_n \rangle \subset \mathcal{H}^1$ converges pointwise to a function h. Then $h \in \mathcal{H}^1$.*

Proof. The topology of pointwise convergence is a Hausdorff topology on the set $\mathcal{H}^1$ supplied with the stronger, compact, ucc-topology. It follows that these topologies are the same on $\mathcal{H}^1$. If a sequence $\langle h_n \rangle \subset \mathcal{H}^1$ converges pointwise to a function h, then by Condition (H_3), there exists a subsequence $\langle h_{n_k} \rangle$ that converges to a function $g \in \mathcal{H}^1$ with respect to the ucc-topology. By our assumption, $g = h$. ☐

4. Feller Compactification

Let $(X, \mathcal{H}, r)$ be a harmonic system, and let $\mathcal{H}_b$ be the set of bounded functions in $\mathcal{H}$. There is a unique (up to homeomorphism) compactification X^F of X such that each function in $\mathcal{H}_b$ extends to the boundary and the set of extensions separates the points of the boundary (see [10] or [18]). We call this compactification the *Feller compactification* of X, and we let Δ_F denote the boundary $X^F \setminus X$. We will use the same notation for a function in $\mathcal{H}_b$ and for its extention to X^F. Recall that $f \vee g$ and $f \wedge g$ denote the pointwise upper and lower envelopes, respectively, of functions f and g. We now show that the *Feller boundary* Δ_F satisfies the following minimum principle.

4.1. **Lemma.** *Fix $h \in \mathcal{H}_b$, and for each $k \in \mathbb{N}$, fix functions h_k and g_k in $\mathcal{H}_b$. For each $n \in \mathbb{N}$, let*

$$p_n := \sum_{k=1}^{n} (h_k \wedge g_k).$$

If $h \geq 0$ on Δ_F, then $h \geq 0$ on X. If $p_n \geq h$ on Δ_F, then $p_n \geq h$ on X.

Proof. If for some $\varepsilon > 0$, $\{x \in X : h + \varepsilon \leq 0\} \neq \emptyset$, then by Condition (H_1), $h(x) \leq -\varepsilon$ for some $x \in \Delta_F$. Therefore, if $h \geq 0$ on Δ_F, $h \geq 0$ on X. Suppose $p_1 \geq h$ on Δ_F. Then $h_1 \geq h$ on Δ_F and therefore on X; similarly $g_1 \geq h$ on X, so $p_1 \geq h$ on X. Suppose we know that for any $g \in \mathcal{H}_b$, if $p_{n-1} \geq g$ on Δ_F, then $p_{n-1} \geq g$ on X. Also assume that on Δ_F we have

$$p_n = (h_n \wedge g_n) + p_{n-1} \geq h.$$

Then on Δ_F,

$$p_{n-1} \geq h - (h_n \wedge g_n) \geq h - h_n \quad \text{and} \quad p_{n-1} \geq h - (h_n \wedge g_n) \geq h - g_n.$$

Therefore on X,

$$p_{n-1} \geq (h - h_n) \vee (h - g_n) = h - (h_n \wedge g_n),$$

and the result follows by induction. □

Given h and g in $\mathcal{H}_b$, we have both the pointwise lower envelope $h \wedge g$ and, since $\mathcal{H}^+ - \mathcal{H}^+$ is a vector lattice, the greatest minorant in $\mathcal{H}_b$ of $h \wedge g$; we denote the latter by $h \wedge_{\mathcal{H}} g$. For the Dirichlet problem, the important part of the Feller boundary Δ_F is a compact subset Γ_F called the *harmonic part* of Δ_F. This set is here defined by setting

$$\Gamma_F := \{z \in \Delta_F : h \wedge g(z) = h \wedge_{\mathcal{H}} g(z) \quad \forall\, h,\, g \in \mathcal{H}_b\}.$$

This subset of Δ_F also satisfies a minimum principle.

4.2. **Proposition.** *If $h \in \mathcal{H}_b$ and $h \geq 0$ on Γ_F, then $h \geq 0$.*

Proof. Given $\varepsilon > 0$, the (possibly empty) set

$$K_\varepsilon := \{z \in \Delta_F : h(z) \leq -\varepsilon\}$$

is compact in $\Delta_F \backslash \Gamma_F$. Hence, there are functions $h_1, \ldots, h_n,\ g_1, \ldots, g_n \in \mathcal{H}_b$ such that

$$p := \sum_{k=1}^{n} (h_k \wedge g_k - h_k \wedge_{\mathcal{H}} g_k)$$

is strictly positive on K_ε. It follows that for some $\alpha \geq 0$, $h + \alpha p + \varepsilon \geq 0$ on Δ_F and therefore on X by Lemma 4.1. It now follows from standard lattice properties that $h + \varepsilon \geq 0$, and so $h \geq 0$ since $\varepsilon > 0$ was arbitrary. □

Given a function f on a set A, we will use $\|f\|_A$ to denote the supremum norm of f when restricted to A. We will write $\mathcal{F}|A$ for the restriction of a class $\mathcal{F}$ of functions to A. As in [21], we have the following relationship between $\mathcal{H}_b$ and $\mathcal{C}(\Gamma_F)$.

4.3. **Proposition.** *The restriction map* $\varphi : h \longrightarrow h|\Gamma_F$ *is an isometric isomorphism from the Banach lattice* $(\mathcal{H}_b, \|\cdot\|_X)$ *onto the Banach lattice* $(\mathcal{C}(\Gamma_F), \|\cdot\|_{\Gamma_F})$.

Proof. It follows easily from Proposition 4.2 that $\|\varphi(h)\|_{\Gamma_F} = \|h\|_X$ for all $h \in \mathcal{H}_b$. The linear space

$$\varphi(\mathcal{H}_b) = \{\varphi(h) : h \in \mathcal{H}_b\}$$

is closed, min-stable, contains the constant functions and, by the defining properties of X^F, separates the points of Γ_F. Thus by the lattice form of the Stone–Weierstrass theorem, $\varphi(\mathcal{H}_b) = \mathcal{C}(\Gamma_F)$. □

We have now shown that for every $f \in \mathcal{C}(\Gamma_F)$ there is a unique function $h_f \in \mathcal{C}(X^F)$ such that $h_f|X \in \mathcal{H}_b$, $h_f|\Gamma_F = f$ and $h_f \geq 0$ if $f \geq 0$. Therefore, for each $z \in X^F$, there is a unique positive Radon measure μ_z^F on Γ_F such that for every $f \in \mathcal{C}(\Gamma_F)$,

$$h_f(z) = \int_{\Gamma_F} f \, d\mu_z^F.$$

Since $1 \in \mathcal{H}$, we have $h_1 = 1$, and therefore μ_z^F is a probability measure.

For every $f \in \mathcal{C}^+(\Gamma_F)$, we set

$$r_F(f) = r(h_f).$$

It follows from the properties of r that the map $f \longrightarrow r_F(f)$ is additive and positive homogenous on $\mathcal{C}^+(\Gamma_F)$. Let $\mathcal{B}(\Gamma_F)$ denote the collection of Borel sets in Γ_F. We may treat r_F as a positive Radon measure on the measurable space $(\Gamma_F, \mathcal{B}(\Gamma_F))$ such that for every $f \in \mathcal{C}(\Gamma_F)$,

$$\int f \, dr_F = r(h_f).$$

The key fact about this measure space is the following theorem.

4.4. **Theorem.** *The space $(\Gamma_F, \mathcal{B}(\Gamma_F), r_F)$ is a perfect measure space. That is, every $L^\infty(\Gamma_F, r_F)$ equivalence class contains a unique representative from $\mathcal{C}(\Gamma_F)$.*

Proof. By Lemma 1.3 of I. Segal [23], we need to show that the following conditions hold.

(i) If U is a nonempty open subset of Γ_F, then $r_F(U) > 0$.
(ii) If $\langle f_n \rangle$ is a decreasing sequence in $\mathcal{C}^+(\Gamma_F)$, then there exists a greatest lower bound f of the sequence in $\mathcal{C}^+(\Gamma_F)$ and

$$\int f\, dr_F = \lim \int f_n\, dr_F.$$

To show that (i) holds, let U be a nonempty open subset of Γ_F, and fix a nonzero $g \in \mathcal{C}^+(\Gamma_F)$ with the support of g contained in U. It follows from Proposition 4.3 that $h_g \neq 0$, and so by Proposition 3.4, $\int g\, dr_F = r(h_g) \neq 0$, whence $r_F(U) > 0$.

To show that (ii) holds, fix a decreasing sequence $\langle f_n \rangle$ in $\mathcal{C}^+(\Gamma_F)$. We may assume that $f_1 \leq 1$. It follows from Proposition 3.5 that the decreasing sequence $\langle h_{f_n} \rangle$ has a limit $h \in \mathcal{H}_b^+$. The function h has a continuous extension to X^F; let f denote the restriction of h to Γ_F. Since $h \leq h_n$ for each n, $f \leq f_n$ for each n. If $g \in \mathcal{C}^+(\Gamma_F)$ is below each f_n, then $h_g \leq h_{f_n}$ for each n, so $h_g \leq h$, whence $g \leq f$. It follows that f is the greatest lower bound of the sequence $\langle f_n \rangle$ in $\mathcal{C}^+(\Gamma_F)$. Since r is continuous on the set $\{h \in \mathcal{H}^+ : h \leq 1\}$ and h_n converges to h uniformly on compact subsets of X, we have

$$\int_{\Gamma_F} f\, dr_F = \int_X h\, dr = \lim_n \int_X h_n\, dr = \lim_n \int_{\Gamma_F} f_n\, dr_F. \ \square$$

Each continuous function f on Δ_F corresponds to a function in $\mathcal{H}_b$, namely $h_{f|\Gamma_F}$. We now have a natural extension of this correspondence.

4.5. **Proposition.** *Let $f : \Delta_F \longrightarrow \mathbb{R}$ be a bounded, Borel measurable function. If for all $x \in X$*

$$h_f(x) := \int_{\Gamma_F} f\, d\mu_x^F,$$

then $h_f \in \mathcal{H}_b$, and if $f \geq 0$,

$$\int_{\Gamma_F} f\, dr_F = r(h_f).$$

Proof. We may assume that $f \geq 0$ and $\int f\, dr_F \leq 1$. First suppose that f is lower semicontinuous. Let $(f_i)_{i\in I} \subset \mathcal{C}(\Delta_F)$ be the upper directed family of all nonnegative continuous functions on Δ_F dominated by f. Since $(h_{f_i})_{i\in I} \subset \mathcal{H}^+$ is directed by increasing order, and by definition

$$\sup_{i\in I} r(h_{f_i}) = \sup_{i\in I} \int f_i dr_F = \int f\, dr_F \leq 1,$$

we have $h_f := \sup h_{f_i} \in \mathcal{H}^1$ by Proposition 3.5. Moreover,

$$\int f\, dr_F = \sup_{i\in I} r(h_{f_i}) = r\left(h_f\right)$$

by the continuity property of r. Now assume that f is just Borel measurable and bounded by a constant M. Let $(g_i)_{i\in I}$ be the lower directed family of all lower semicontinuous functions on Δ_F that dominate f and are dominated by M. Since $(h_{g_i})_{i\in I} \subset \mathcal{H}^+$ is uniformly bounded by M and directed by decreasing order, it follows from Proposition 3.5 that $h_f := \inf h_{g_i} \in \mathcal{H}^1$. Moreover, by definition and the continuity of r,

$$\int f\, dr_F = \inf_{i\in I} \int g_i dr_F = \inf_{i\in I} r(h_{g_i}) = r\left(h_f\right). \ \square$$

4.6. **Proposition.** *For every $x \in X$ there is a unique continuous function h_x on X^F with $h_x|X \in \mathcal{H}_b^+$ and $\mu_x^F = h_x r_F$ on Γ_F.*

Proof. Fix $x \in X$; we show first that μ_x^F is absolutely continuous with respect to r_F. Given a Borel subset A of Γ_F with $r_F(A) = 0$, consider the function $h_A := h_{1_A}$. By Proposition 4.5, $h_A \in \mathcal{H}_b^+$ and $r(h_A) = r_F(A) = 0$, whence $h_A = 0$ by Proposition 3.4. It follows that $\mu_x^F(A) = h_A(x) = 0$, and thus $\mu_x^F \ll r_F$.

Let f_x be a representative of the Radon–Nikodým derivative $\frac{d\mu_x^F}{dr_F}$. To show that $f_x \in L^\infty(\Gamma_F, r_F)$, we fix for each $m \in \mathbb{N}$ the set

$$A_m := \{y \in \Gamma_F : f_x(y) \geq 2^m\}.$$

For each $m \in \mathbb{N}$, we set

$$f_m = (2^m r_F(A_m))^{-1} \cdot 1_{A_m}.$$

if $r_F(A_m) > 0$, and otherwise we set $f_m = 0$. Each function f_m is bounded and has an extension $h_m \in \mathcal{H}_b^+$. By Proposition 4.5, for all $n \in \mathbb{N}$

$$r\left(\sum_{m=1}^n h_m\right) = \sum_{m=1}^n r(h_m) = \sum_{m=1}^n \int f_m dr_F \leq \sum_{m=1}^n 2^{-m} \leq 1,$$

so $\sum_{m=1}^{\infty} h_m \in \mathcal{H}^+$ by Proposition 3.5; in particular $\sum_{m=1}^{\infty} h_m(x) < +\infty$. On the other hand, for each $m \in \mathbb{N}$ for which $f_m \neq 0$,

$$h_m(x) = \int f_m d\mu_x^F = \int f_m f_x dr_F = (2^m r_F(A_m))^{-1} \int 1_{A_m} f_x dr_F \geq 1,$$

whence there is a maximum m for which $r_F(A_m) > 0$. It follows that f_x is bounded, and so we may assume that f_x is a continuous function on Γ_F. This continuous function has a continuous extension h_x on X^F with $h_x|X \in \mathcal{H}_b^+$. □

5. A Martin-type Compactification

The Feller compactification of X is a large compactification, roughly like the Stone-Čech compactification. In this section, we project X^F onto a metrizable compactification that is similar to, and will often be equivalent to, the Martin compactification of X when the latter compactification is defined.

Let $\mathcal{Q} := \{h_x : x \in X\}$. Here, for every $x \in X$, h_x is the unique continuous function on X^F such that $h_x|X \in \mathcal{H}_b^+$ and $\mu_x^F = h_x r_F$; the existence is assured by Proposition 4.6. Now, there is a unique (up to homeomorphism) compactification of X, called the $\mathcal{Q}$-compactification, such that each function in $\mathcal{Q}$ extends to the boundary of X and the set of extensions separates the points of the boundary (see [10] or [18]).

For this family $\mathcal{Q}$, the compactification of X will be denoted by $\widehat{X}$, and the boundary by

$$\Delta := \widehat{X} \setminus X.$$

Since $Q|X \subset \mathcal{H}_b^+$, there exists a continuous surjection $\Phi : X^F \longrightarrow \widehat{X}$ such that

(i) $\Phi(x) = x$ for every $x \in X$, $\Phi(y) \in \Delta$ for every $y \in \Delta_F$, and
(ii) if $y, z \in \Delta_F$ then $\Phi(y) = \Phi(z)$ if and only if $h_x(y) = h_x(z)$ for every $x \in X$.

As usual, we may use Φ to project a measure ν from X^F to $\widehat{X}$; i.e., $\Phi(\nu)(A) = \nu\left(\Phi^{-1}[A]\right)$ for every measurable set A.

5.1. **Definition.** *For each $x \in X$, we set*

$$\mu_x := \Phi\left(\mu_x^F\right), \quad \sigma := \Phi\left(r_F\right), \quad \text{and} \quad \Sigma := \text{support}(\sigma) \subset \Delta.$$

We define a kernel $Q : X \times \widehat{X} \longrightarrow \mathbb{R}^+$ by setting

$$Q(x, z) := h_x(z),\ (x, z) \in X \times \widehat{X}.$$

5.2. **Proposition.** *The kernel Q has the following properties for every $x, y \in X$:*

(1) $Q(x,y) = Q(y,x)$;

(2) $Q(x,\cdot) \in \mathcal{C}\left(\widehat{X}\right)$;

(3) $Q(x,\cdot)\,|\,X \in \mathcal{H}_b^+$;

(4) $\mu_x = Q(x,\cdot)\sigma$;

(5) $Q(x,y) = \int Q(x,z)Q(y,z)\,\sigma(dz)$;

(6) $r(Q(x,\cdot)\,|\,X) = 1$. *Moreover,*

(7) $Q(\cdot,z) \in \mathcal{H}^1$ *for all* $z \in \Delta$.

Proof.

(1) $Q(x,y) = h_x(y) = \int \frac{d\mu_x^F}{dr_F} \cdot \frac{d\mu_y^F}{dr_F}\, dr_F = Q(y,x)$.

(2), (3) These properties follow from the definitions.

(4) For every $y \in \Delta_F$, $Q(x,\Phi(y)) = h_x(y)$. Therefore, for every $f \in \mathcal{C}(\Delta)$, we have

$$\int_\Delta f d\mu_x = \int_{\Delta_F} f \circ \Phi\, d\mu_x^F = \int_{\Delta_F} (f \circ \Phi) \cdot h_x\, dr_F = \int_\Delta f \cdot Q(x,\cdot)\, d\sigma.$$

(5)

$$Q(x,y) = h_x(y) = \int_{\Delta_F} h_x(z)h_y(z)\, r_F(dz) = \int_\Delta Q(x,z)Q(y,z)\,\sigma(dz).$$

(6)

$$r\left(Q(x,\cdot)|X\right) = r\left(h_x|X\right) = \int_{\Delta_F} h_x\, dr_F = \int_{\Delta_F} 1 d\mu_x^F = h_1(x) = 1.$$

(7) Let $\langle x_n \rangle \subset X$ be a sequence such that $\lim x_n = z$. Since for every $x \in X$

$$Q(x,z) = \lim Q(x,x_n) = \lim Q(x_n,x),$$

$Q(\cdot,z)$ is by (6) the pointwise limit of a sequence of functions in $\mathcal{H}^1$, whence by Proposition 3.6, $Q(\cdot,z) \in \mathcal{H}^1$, and in particular $r(Q(\cdot,z)) \leq 1$. □

5.3. **Proposition.** *Let* $f : \Delta \longrightarrow \mathbb{R}$ *be a bounded, Borel measurable function. If for all* $x \in X$

$$H_f(x) := \int_\Delta f\, d\mu_x,$$

then $H_f \in \mathcal{H}_b$, *and if* $f \geq 0$,

$$\int_\Delta f\, d\sigma = r(H_f).$$

Proof. The function $f \circ \Phi$ is a bounded Borel measurable function on Δ_F. Moreover, for all $x \in X$,

$$H_f(x) = \int_\Delta f\, d\mu_x = \int_{\Delta_F} f \circ \Phi(z)\, \mu_x^F(dz).$$

Therefore by Proposition 4.5, $H_f \in \mathcal{H}_b$, and if $f \geq 0$,

$$r(H_f) = \int_{\Delta_F} f \circ \Phi(z)\, r_F(dz) = \int_\Delta f\, d\sigma. \ \square$$

5.4. **Proposition.** *The mapping* $T : y \longmapsto Q(\cdot, y)$ *from* $\widehat{X}$ *into* $\mathcal{H}^1$ *is continuous. Moreover,* $T|\Delta$ *is injective; in particular,* $T|\Delta$ *is a homeomorphism from* Δ *into* $\mathcal{H}^1$.

Proof. Define $T : X \longrightarrow \mathcal{H}^1$ by

$$T(x) = Q(\cdot, x), \quad x \in X.$$

Given $x, y_1, \ldots, y_m \in X$ and $\varepsilon > 0$ there is, by (2) of Proposition 5.2, a neighborhood U of x such that for all $x' \in U$,

$$|Q(y_i, x') - Q(y_i, x)| < \varepsilon,\ 1 \leq i \leq m.$$

It follows that T is continuous with respect to the topology of pointwise convergence for $\mathcal{H}^1$, and by Proposition 3.6, this topology is the same as the topology of uniform convergence on compact sets. Since $\mathcal{H}^1$ is compact in this topology, there is a compactification $\widetilde{X}$ of X such that T extends continuously to $\widetilde{X}$ and separates the points of $\widetilde{X} \setminus X$. For each $z \in \widetilde{X}$ let $\widetilde{Q}(\cdot, z)$ denote the function $T(z)$. Given $x \in X$, $\widetilde{Q}(x, \cdot)$ is a continuous extension to $\widetilde{X}$ of $Q(x, \cdot)$, for if $z \in \widetilde{X} \setminus X$ and $\varepsilon > 0$ are fixed, then there is a neighborhood U of z such that for all $z' \in U$

$$\left| \widetilde{Q}(x, z') - \widetilde{Q}(x, z) \right| < \varepsilon.$$

If z_1, z_2 are points in $\widetilde{X} \setminus X$ such that $z_1 \neq z_2$, then for some $x \in X$

$$\widetilde{Q}(x, z_1) \neq \widetilde{Q}(x, z_2).$$

It now follows from the definition of $\widehat{X}$ that we may take $\widetilde{X} = \widehat{X}$. Therefore, the mapping $T : z \longmapsto Q(\cdot, z)$ is continuous on $\widehat{X}$ and injective on Δ. Since Δ is compact, $T|\Delta : \Delta \longrightarrow \mathcal{H}^1$ is a homeomorphism. $\square$

5.5. **Remark.** If T is injective on $\widehat{X}$, e.g., if $\mathcal{H}_b$ separates the points of X and for each $z \in \Delta$ and $x \in X$, $Q(\cdot, z) \neq Q(\cdot, x)$, then $\widehat{X}$ is metrizable. In any case, the boundary Δ is metrizable.

In what follows, we shall identify Δ with $T(\Delta) \subset \mathcal{H}^1$.

6. Integral Representation of Bounded Functions

We have seen in Proposition 5.3 that every bounded, Borel measurable function f on Δ corresponds to a function $H_f \in \mathcal{H}_b$ given by setting

$$H_f(x) := \int_\Delta f \, d\mu_x.$$

To extend this representation, we set

$$\mathcal{E}^1 := \mathcal{E} \setminus \{0\}, \text{ and}$$
$$\Delta_e := \Delta \cap \mathcal{E}^1.$$

Note that Δ_e can be considered as either a subset of $\mathcal{H}^1$ or of Δ. It is a G_δ-set in either case. We will use facts from Section 3 about abstract integral representations of elements of $\mathcal{H}^1$.

6.1. **Definition.** *For every $h \in \mathcal{H}_b^+$, we define a measure m_h on Δ by setting $m_h := \Phi(hr_F)$. This means that for all $f \in C(\Delta)$,*

$$\int f \, dm_h := \int f \circ \Phi \cdot h dr_F.$$

6.2. **Lemma.** *If $h \in \mathcal{H}_b^+$ with $r(h) = 1$ then m_h represents h in the Choquet simplex $\mathcal{H}^1$.*

Proof. Since

$$\int 1 dm_h = \int h \, dr_F = r(h) = 1,$$

m_h is a probability measure on $\mathcal{H}^1$; as such, it represents a unique function $h' \in \mathcal{H}^1$. To show that $h' = h$, we note that for every $x \in X$,

$$\begin{aligned} h'(x) &= \int Q(x,z) \, m_h(dz) = \int Q(x, \Phi(y)) h(y) r_F(dy) \\ &= \int h_x(y) h(y) r_F(dy) = \int h(y) \mu_x^F(dy) \\ &= h(x). \ \square \end{aligned}$$

6.3. **Theorem.** *If $h \in \mathcal{H}_b^+$ and $r(h) = 1$, then m_h is the unique minimal representing measure μ_h for h defined on $\mathcal{H}^1$. Furthermore, there exists an $f \geq 0$ in $L^\infty(\sigma)$ such that $\mu_h = f\sigma$.*

Proof. The mapping $h \longmapsto m_h$ is clearly affine on the set $\mathcal{H}_b^1 = \{h \in \mathcal{H}_b^+ : r(h) = 1\}$. Fix $h \in \mathcal{H}_b^1$, and let λ be a probability measure on $\mathcal{H}^1$ that represents h. Furthermore, let $\sum_{i=1}^m \alpha_i \lambda_i$ be an affine decomposition of λ. Each λ_i represents some nonnegative $h_i \in \mathcal{H}^1$. Therefore, $h = \sum\limits_{i=1}^{m} \alpha_i h_i$ since for every $x \in X$

$$h(x) = \int\limits_{\mathcal{H}^1} k(x)\lambda(dk) = \sum_{i=1}^{m} \alpha_i \int\limits_{\mathcal{H}^1} k(x)\lambda_i(dk) = \sum_{i=1}^{m} \alpha_i h_i(x).$$

We conclude that $h_i \in \mathcal{H}_b^1$ for all i with $1 \leq i \leq m$. Moreover, $m_h = \sum\limits_{i=1}^{m} \alpha_i m_{h_i}$ where m_{h_i} represents h_i for all i with $1 \leq i \leq m$. We now apply a theorem of Cartier, Fell, and Meyer. (See Proposition 4.2 in [19]; also see the corollary there due to B. Fuchssteiner.) From this theorem, it follows that m_h is the unique representing measure μ_h that is maximal with respect to the Choquet ordering.

It remains to show that for each $h \in \mathcal{H}_b^1$, μ_h is absolutely continuous with respect to σ. For this purpose let $A \subset \Delta$ be a Borel set such that $\sigma(A) = 0$. Then

$$r_F\left(\Phi^{-1}[A]\right) = \sigma(A) = 0,$$

and therefore

$$\mu_h(A) = \int 1_A \circ \Phi \cdot h\, dr_F = 0.$$

Thus we have $\mu_h = f\sigma$ with $f \geq 0$ in $L^1(\sigma)$. The mapping $g \longmapsto \mu_g$ is order preserving, and so $f \in L^\infty(\sigma)$. $\square$

6.4. **Remark.** (1) The measure σ is the representing measure μ_1 for 1. Indeed, since by Theorem 6.3 $\mu_1 = m_1$, we have for every $f \in \mathcal{C}(\Delta)$,

$$\int f d\mu_1 = \int f \circ \Phi\, dr_F = \int f d\sigma.$$

(2) The set $\Sigma :=$ support(σ) is equal to $\overline{\Delta}_e$.

(3) It follows from (2) above and II.62 of [8] that for σ-almost all $z \in \Delta$, $r(Q(\cdot, z)) = 1$.

(4) Fix $h > 0$ in $\mathcal{H}_b$. Then $r(h) \neq 0$ and $h' := \frac{1}{r(h)} h \in \mathcal{H}_b^1$. Hence

$$\mu_h = r(h)\mu_{h'},$$

and h can be represented by a unique measure μ_h on Δ such that $\mu_h(\Delta \backslash \Delta_e) = 0$. Moreover,

$$\mu_h(\Delta) = r(h) \cdot \int_\Delta 1 d\mu_{h'} = r(h) \cdot \int_{\Delta_F} h' dr_F = r(h).$$

(5) It follows from (1) above and Proposition 5.2 that for all $x, y \in X$,

$$Q(x,y) = \int_\Delta Q(x,z)Q(y,z)\,\sigma(dz) = \int_{\mathcal{E}^1} g(x)g(y)\,\mu_1(dg).$$

6.5. **Example.** *Harmonic Functions on the Unit Ball or on a Lipschitz Domain*
Let X be the open unit ball in $\mathbb{R}^n$, and let $S = \partial X = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Let $\mathcal{H} = \{h \in \mathcal{C}^2(X) : \Delta h = 0\}$, and let r be unit mass at 0. Then $(X, \mathcal{H}, r)$ is a harmonic system. For any $z \in S$, let

$$P(x,z) := \frac{1 - \|x\|^2}{\|x - z\|^n}, \quad x \in X$$

be the Poisson kernel. It is will known that the functions $P(\cdot, z),\ z \in S$ are exactly the minimal harmonic functions with

$$\int P(y,z)\,r(dy) = P(0,z) = 1.$$

Hence for all $x, y \in X$,

$$Q(x,y) = \int P(x,z)P(y,z)\,\sigma(dz)$$

where $\sigma = \mu_1$ is normalized surface measure on S. On the other hand, the Poisson formula gives a solution to the Dirichlet problem. Using H_f to denote the solution for a boundary function f, we have for all $z \in S$,

$$\lim_{y \to z} Q(x,y) = \lim_{y \to z} H_{Q(x,\cdot)}(y) = P(x,z).$$

Since the set $\{P(x, \cdot) : x \in X\}$ separates the points of $S = \partial X$, the closed unit ball $\overline{X} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is just our compactification $\widehat{X}$ with $\Delta = \partial X$. As noted in [19], it follows from results of Hunt and Wheeden [13] that for ordinary harmonic functions on a Lipschitz domain X in $\mathbb{R}^n$, the compactification $\widehat{X}$ with its association between boundary points and points of $\mathcal{H}^1$ is the same as the Martin compactification.

6.6. **Example.** *Harmonic Function on the Punctured Unit Ball*
Now let $X := \{x \in \mathbb{R}^n : 0 < \|x\| < 1\}$, $\mathcal{H} = \{h \in \mathcal{C}^2(X) : \Delta h = 0\}$, and let r be unit mass at some point $a \in X$. Then $(X, \mathcal{H}, r)$ is again a harmonic system. As in the last example,

$$\widehat{X} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

In this case, however, $Q(\cdot, 0)$ is a positive, bounded harmonic function on X. This example shows that even if the compactifications are homeomorphic, the compactification $\widehat{X}$ may be different from the Martin compactification in terms of the association of boundary points with functions in $\mathcal{H}^1$.

7. Dirichlet Problem

Let $f : \Delta \longrightarrow \mathbb{R}$ be a bounded, Borel measurable function. We have seen in Proposition 5.3 that setting

$$H_f(x) = \int_\Delta f \, d\mu_x = \int_\Delta f(z) Q(x, z) \, \sigma(dz)$$

for each $x \in X$ produces an element of $\mathcal{H}_b$. When $f \geq 0$, it follows from Theorem 6.3 and the uniqueness of Radon–Nikodým derivatives that

$$\mu_{H_f} = f\sigma.$$

In this section, we consider the behavior at the boundary Δ of the functions H_f when f is continuous on Δ.

7.1. **Definition.** *As usual, a point $z \in \Delta$ is called **regular** if for every $f \in \mathcal{C}(\Delta)$*

$$\lim_{x \to z} H_f(x) = f(z),$$

i.e., $\lim_{x \to z} \mu_x = \varepsilon_z$ in the vague (a.k.a. weak) topology. We let Δ_{reg} denote the set of regular points in Δ.*

7.2. **Proposition.** *We have $\Delta_e \subset \Delta_{\text{reg}} \subset \Sigma = \overline{\Delta}_e$.*

Proof. Fix $z \in \Delta_e$. Let $\mathcal{F}$ be a filter on X such that $\lim \mathcal{F} = z$. Let $\mathcal{G}$ be an ultrafilter on X finer than $\mathcal{F}$. Then $\lim_{x, \mathcal{G}} \mu_x = \nu$ exists since the set of probability measures on Δ is vaguely compact. We must show that $\nu = \varepsilon_z$. The measure ν represents the function $Q(\cdot, z)$ on $\mathcal{H}^1$. Indeed, for any $x' \in X$,

$$\int Q(x', y) \nu(dy) = \lim_{x, \mathcal{G}} \int Q(x', y) \mu_x(dy) = \lim_{x, \mathcal{G}} Q(x', x) = Q(x', z).$$

Since ε_z is the only probability measure on $\mathcal{H}^1$ that represents $Q(\cdot, z)$, we must have $\nu = \varepsilon_z$. Therefore, $z \in \Delta_{\text{reg}}$. The rest follows from the definition of regular points and Remark 6.4(2). □

7.3. **Corollary.** *σ-almost all points of Δ are regular.*

7.4. **Definition.** *A point $z \in \Delta \setminus \Delta_{\text{reg}}$ is called **semiregular**, if we have a limit $\lim_{x \in X, \, x \to z} H_f(x)$ for all $f \in \mathcal{C}(\Delta)$. We denote by Δ_{sem} the set of all semiregular points in Δ.*

7.5. **Proposition.** *The set $\Delta_{\text{sem}} \subset \Delta \setminus \Sigma$.*

Proof. Fix $z \in \Delta_{\mathrm{sem}}$. There exists a probability measure μ on Δ such that for all $g \in \mathcal{C}(\Delta)$

$$\int g d\mu = \lim_{x \in X,\, x \to z} H_g(x).$$

Since z is not regular there is a function $f \in \mathcal{C}(\Delta)$ such that $\int f d\mu \neq f(z)$. Let $\varepsilon := \frac{1}{2}\left|\int f d\mu - f(z)\right| > 0$. Assume $z \in \Sigma$, and let $\langle z_n \rangle$ be a sequence in Δ_{reg} such that $\lim_{n\to\infty} z_n = z$. There exist an $n_0 \in \mathbb{N}$ and a neighborhood U of z in $\widehat{X}$ such that for all $n \geq n_0$ and all $x \in U \cap X$ we have

$$z_n \in U, \quad |f(z) - f(z_n)| < \frac{\varepsilon}{3}, \quad \left|\int f d\mu - H_f(x)\right| < \frac{\varepsilon}{3}.$$

Fix $n \geq n_0$, and choose a neighborhood V of z_n such that $V \subset U$ and for all $x \in V \cap X$,

$$|H_f(x) - f(z_n)| < \frac{\varepsilon}{3}.$$

The following inequality for $x \in V \cap X$ gives the desired contradiction:

$$\begin{aligned} 2\varepsilon &= \left|\int f d\mu - f(z)\right| \\ &\leq \left|\int f d\mu - H_f(x)\right| + |H_f(x) - f(z_n)| + |f(z_n) - f(z)| < \varepsilon. \quad \square \end{aligned}$$

7.6. **Proposition.** *Suppose* $\Delta_e = \Sigma$ *. Then for all* $f \in \mathcal{C}(\Delta)$ *and all* $z_0 \in \Delta$*,*

$$\lim_{x \in X,\, x \to z_0} H_f(x) = \int f\, d\mu_{Q(\cdot, z_0)},$$

where $\mu_{Q(\cdot,z_0)}$ *is the unique minimal representing measure for* $Q(\cdot, z_0)$ *defined on* $\mathcal{H}^1$*. It follows that in this case,* $\Delta_{\mathrm{sem}} = \Delta \setminus \Sigma$*.*

Proof. Fix $z_0 \in \Delta \setminus \Sigma$, and let $\mathcal{F}$ be an ultrafilter on X with $\lim \mathcal{F} = z_0$. There exists a probability measure μ on Δ such that for all $f \in \mathcal{C}(\Delta)$

$$\int f d\mu = \lim\nolimits_{\mathcal{F}} H_f(x)$$

Since H_f depends only on $f|\Sigma$, the measure μ is supported by Σ; moreover it represents $Q(\cdot, z_0)$. Indeed, if $y \in X$ and $f = Q(y, \cdot)$ then

$$\int Q(y,z)\mu(dz) = \lim\nolimits_{\mathcal{F}} H_{Q(y,\cdot)} = \lim\nolimits_{\mathcal{F}} Q(y, \cdot) = Q(y, z_0).$$

The proposition follows from the fact that for any choice of the ultrafilter $\mathcal{F}$, the measure μ is the minimal representing measure $\mu_{Q(\cdot,z_0)}$. $\square$

If the kernel Q is very tame at a regular or semiregular point of Δ, then as we now show, there is a limit at the point for all bounded h in $\mathcal{H}$.

7.7. **Proposition.** *Fix* $z_0 \in \Delta_{\text{reg}} \cup \Delta_{\text{sem}}$, *and suppose that*

$$\alpha := \limsup_{x\in X,\ x\to z_0} Q(x,x) < \infty.$$

Then $\lim_{x\in X,\ x\to z_0} h(x)$ *exists for all* $h \in \mathcal{H}_b$.

Proof. Fix $\varepsilon > 0$ and $h \in \mathcal{H}_b$. For some $f \in L^\infty(\sigma) \subset L^2(\sigma)$ and all $x \in X$,

$$h(x) = \int Q(x,z) f(z)\, \sigma(dz).$$

Fix $g \in \mathcal{C}(\Delta)$ with $\|f-g\|_2 \le \varepsilon/\sqrt{1+\alpha}$. Now for all $x \in X$,

$$\begin{aligned} |h(x) - H_g(x)| &\le \int |f(z) - g(z)|\, Q(x,z)\sigma(dz) \le \|f-g\|_2 \|Q(x,\cdot)\|_2 \\ &\le \frac{\varepsilon}{\sqrt{1+\alpha}} \cdot Q(x,x)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\limsup_{x\in X,\ x\to z_0} |h(x) - H_g(x)| \le \frac{\varepsilon\sqrt{\alpha}}{\sqrt{1+\alpha}} \le \varepsilon.$$

Let $\beta := \lim_{x\to z_0} H_g(x)$. Then

$$\begin{aligned} \beta - \varepsilon = \lim_{x\in X, x\to z_0} H_{g-\varepsilon}(x) &\le \liminf_{x\in X,\ x\to z_0} h(x) \\ &\le \limsup_{x\in X,\ x\to z_0} h(x) \le \lim_{x\in X, x\to z_0} H_{g+\varepsilon}(x) = \beta + \varepsilon, \end{aligned}$$

whence

$$0 \le \limsup_{x\in X,\ x\to z_0} h(x) - \liminf_{x\in X,\ x\to z_0} h(x) \le 2\varepsilon. \ \square$$

7.8. **Remark.** (added March, 2001) After completing this paper, the authors received a preprint [16] from Teruo Ikegami and Masaharu Nishio extending the results in [19] and [20], from which we too have proceeded. Instead of the Feller compactification, however, they start with an arbitrary metrizable and resolutive compactification. Of the new results in the two papers, the principal overlap is in the investigation of semiregular points.

8. r-Sturdy Functions

Recall that for our harmonic system, the set $\mathcal{H}^1 := \{h \in \mathcal{H}^+ : r(h) \leq 1\}$ is supplied with the ucc-topology. We set $\widetilde{\mathcal{Q}}$ equal to the closed convex cone in $\mathcal{H}^+$ generated by $\mathcal{Q} := \{Q(\cdot, z) : z \in \Delta\}$, and we set $\mathcal{Q}^1 := \mathcal{H}^1 \cap \widetilde{\mathcal{Q}}$.

This section is focused on the properties of the convex set $\mathcal{H}^1_\Sigma$; it is the closure in $\mathcal{H}^1$ of $\mathcal{H}^1_b = \mathcal{H}^1 \cap \mathcal{H}_b$. We have given the name *r-sturdy functions* to the elements of this closed convex subset of $\mathcal{H}^1$. It will be shown that it is precisely the r-sturdy functions that are given by representing measures on the boundary Δ.

8.1. **Remark.**

(1) The boundary $\Delta \subset \mathcal{H}^1_\Sigma$ since for each $z_0 \in \Delta$,
$$Q(\cdot, z_0) = \lim_{x \in X,\, x \to z_0} Q(\cdot, x).$$

(2) If $(X, \mathcal{H})$ is a harmonic space, then by Proposition 2.3 a positive harmonic function h on X is sturdy if and only if h is r-sturdy for some reference measure r on X.

8.2. **Theorem.** *Given $h \in \mathcal{H}^1$, the following assertions are equivalent:*

(1) *The function h is r-sturdy.*

(2) *There exists a nonnegative measure μ on Σ with $\mu(\Sigma) \leq 1$ such that*
$$h(x) = \int Q(x, z)\mu(dz) \quad \text{for all} \quad x \in X.$$

(3) *There exists a nonnegative measure μ on Δ with $\mu(\Delta) \leq 1$ such that*
$$h(x) = \int Q(x, z)\mu(dz) \quad \text{for all} \quad x \in X.$$

(4) *The function h is in $\mathcal{Q}^1$.*

Proof. (1) $\Rightarrow$ (2): Let $\langle h_n \rangle$ be a sequence in $\mathcal{H}^1_b$ converging to h. By Theorem 6.3 the measures μ_{h_n} are absolutely continuous with respect to σ, and by Proposition 5.3, for each $n \in \mathbb{N}$, $\mu_{h_n}(\Delta) = r(h_n) \leq 1$. We may, therefore, assume that the sequence $\langle \mu_{h_n} \rangle$ converges vaguely to some nonnegative measure μ on Δ with $\mu(\Delta) \leq 1$ and support $\mu \subset \Sigma$. It follows that for all $x \in X$,
$$\int Q(x, z)\mu(dz) = \lim_n \int Q(x, z)\mu_{h_n}(dz) = \lim_n h_n(x) = h(x).$$

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (4): If $\mu = 0$, the result is trivial. Therefore, we may assume that μ is a probability measure. Now h is the barycenter of a probability measure on $\Delta \subset \mathcal{Q}^1$, whence $h \in \mathcal{Q}^1$.

(4) $\Rightarrow$ (1): Since $\mathcal{Q} \subset \overline{\mathcal{H}_b^1} = \mathcal{H}_\Sigma^1$, we have $\mathcal{Q}^1 \subset \mathcal{H}_\Sigma^1$. □

8.3. **Corollary.** *If $\mathcal{E}^1 \subset \Sigma$ then $\mathcal{H}^1 = \mathcal{H}_\Sigma^1$, i.e., every $h \in \mathcal{H}^1$ is r-sturdy.*

8.4. **Corollary.** *The set $\Delta_e = \mathcal{E}^1 \cap \mathcal{H}_\Sigma^1$.*

Proof. $\Delta_e = \mathcal{E}^1 \cap \Delta \subset \mathcal{E}^1 \cap \mathcal{H}_\Sigma^1 = \mathcal{E}^1 \cap \mathcal{Q}^1 \subset \Delta_e$. □

A subclass of the r-sturdy functions is formed by those functions in $\mathcal{H}^1$ that are the limit of an increasing sequence of functions in $\mathcal{H}_b^1$. These are called the *quasibounded* elements of $\mathcal{H}^1$.

8.5. **Proposition.** *Fix h in $\mathcal{H}^1$. The following assertions are equivalent:*

(1) *There exists a function $f \geq 0$ in $L^1(\sigma)$ such that*

$$h(x) = \int Q(x,z) f(z)\, \sigma(dz) \quad \text{for all} \quad x \in X.$$

(2) *h is quasi-bounded.*

Proof. (1) $\Rightarrow$ (2): For each $n \in \mathbb{N}$, set

$$h_n(x) := \int Q(x,z)(f \wedge n)\sigma(dz), \ x \in X.$$

By the obvious order relations and Proposition 5.3, $\langle h_n \rangle \subset \mathcal{H}_b^1$ and the h_n's increase to h.

(2) $\Rightarrow$ (1): This follows from Theorem 6.3 and the Monotone Convergence Theorem. □

8.6. **Example.** *Heat equation*

Let $\mathcal{H}$ be the space of solutions of the heat equation

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \quad \text{on} \quad X := \mathbb{R} \times]0, +\infty[$$

Let λ denote the restriction of Lebesgue measure on $\mathbb{R}$ and set

$$f(t) := \begin{cases} 0 & \text{if } 0 < t \leq \frac{1}{4} \\ \frac{1}{4}\, t^{-3/2} & \text{if } \frac{1}{4} < t < +\infty. \end{cases}$$

Then by K. Janßen [17],

$$r := \varepsilon_0 \otimes (f\lambda)$$

is a reference measure, i.e. $(X, \mathcal{H}, r)$ is a harmonic system. According to M. Sieveking [24], every function on an extreme ray of $\mathcal{H}^+$ is given by

$$(x,t) \longmapsto c\,t^{-1/2} \exp\left(-\frac{(x-a)^2}{4t}\right) \ , \ (x,t) \in X, \ a \in \mathbb{R}, \ c \in \mathbb{R}^+$$

Therefore, after normalization,

$$\mathcal{E}^1 = \{h_a : a \in \mathbb{R}\},$$

where for $a \in \mathbb{R}$ and $(x,t) \in X$,

$$h_a(x,t) = \begin{cases} \frac{a^2}{1-\exp(-a^2)} \cdot t^{-1/2} \exp\left(\frac{(x-a)^2}{4t}\right) & \text{if } a \neq 0 \\ t^{-1/2}\exp\left(-\frac{x^2}{4t}\right) & \text{if } a = 0. \end{cases}$$

By K. Janßen [17] the minimal representing measure μ_1 of the constant function 1 is given by

$$\mu_1 := g\,\lambda$$

where

$$g(a) := \frac{1-\exp(a^2)}{2\sqrt{\pi}a^2}, \quad a \in \mathbb{R}.$$

That is, μ_1 has a strictly positive continuous density with respect to Lebesgue measure λ, whence $\mathcal{E}^1 \subset \Sigma$. It now follows from Corollary 8.3 that all positive functions in $\mathcal{H}$ are sturdy.

By Theorem 8.2, an r-sturdy function h on X admits an integral representation with respect to the kernel Q. One may ask if it is possible to get such an integral representation using the minimal representing measure μ_h for h. Since $\mathcal{H}^1_\Sigma$ is a convex set, this property will hold for all r-sturdy functions if and only if $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$. Although $\mathcal{H}^1_b$ is certainly a face of $\mathcal{H}^1$, its closure $\mathcal{H}^1_\Sigma = \overline{\mathcal{H}^1_b}$ is in general not a face. For a counterexample, see Example 10.2, (3). In what follows, we shall present a number of equivalent conditions for the fundamental property that $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$.

8.7. **Proposition.** *The following conditions are equivalent:*

(1) $\mathcal{H}^1_\Sigma$ *is a face of* $\mathcal{H}^1$.

(2) *For every* $z_0 \in \Sigma \setminus \Delta_e$ *there exists a positive measure* μ *on* Δ *with* support $\mu \subset \Sigma$ *such that* μ *is not unit mass* ε_{z_0} *at* z_0, *but*

$$Q(\cdot, z_0) = \int Q(\cdot, z)\mu(dz).$$

(3) *For every* $h \in \mathcal{H}^1_\Sigma$ *the minimal representing measure* μ_h *satisfies*

$$\text{support}\,\mu_h \subset \Sigma.$$

Proof. (1) $\Rightarrow$ (2): If $z_0 \in \Sigma \setminus \Delta_e$ then the minimal representing measure of $Q(\cdot, z_0)$ is supported by $\mathcal{E}^1 \cap \mathcal{H}^1_\Sigma = \Delta_e$, hence that measure is different from ε_{z_0}.

(2) $\Rightarrow$ (1): By assumption, no point of $\Sigma \setminus \Delta_e$ is extreme in $\mathcal{H}^1_\Sigma$. Therefore, the representing measures on the extreme points of $\mathcal{H}^1_\Sigma$ are minimal representing measures, whence $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$.

(1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. □

8.8. **Corollary.** *If $\Delta_e = \Delta_{\text{reg}}$ then $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$.*

Proof. Fix $z_0 \in \Sigma \setminus \Delta_e = \Sigma \setminus \Delta_{\text{reg}}$. There exists an ultrafilter $\mathcal{F}$ on X such that $\lim \mathcal{F} = z_0$ and $\lim\limits_{x, \mathcal{F}} \mu_x = \nu \neq \varepsilon_{z_0}$. As in proof of Proposition 8.2, the measure ν represents $Q(\cdot, z_0)$. The assertion now follows from Proposition 8.7. □

8.9. **Remark.** Suppose Δ_e is closed, so $\Delta_e = \Sigma$. Then $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$ since

$$\Sigma = \Delta_e \subset \Delta_{\text{reg}} \subset \Sigma, \text{ i.e., } \Delta_e = \Delta_{\text{reg}}.$$

8.10. **Remark.** Part (3) of Example 10.2 shows that H^1_Σ is in general not a face. If $(X, \mathcal{H})$ is a harmonic space then Proposition 2.3 implies that the convex cone of sturdy functions is a hereditary subcone of $\mathcal{H}^+$ if and only if $\mathcal{H}^1_\Sigma$ for r is a face of $\mathcal{H}^1$ for every reference measure r on X.

8.11. **Example.** An application of Theorem 8.2 is the fact that for classical harmonic functions on the unit Ball in $\mathbb{R}^n$, all positive harmonic functions are sturdy, whereas the harmonic function

$$x \longmapsto -\log \|x\|$$

on the punctured unit disk in $\mathbb{R}^2$ is not sturdy. (Also see Examples 6.5 and 6.6).

9. Simpliciality

Another characterization for the case that $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$ uses the function space $H(\Delta)$. That is, we start by setting

$$H\left(\widehat{X}\right) := \{h \in \mathcal{C}\left(\widehat{X}\right) : h|X \in \mathcal{H}^1\},$$

and then set

$$H(\Delta) := H\left(\widehat{X}\right)|\Delta.$$

9.1. **Definition.** *The Choquet boundary of $H(\Delta)$ is denoted by $\partial_e\Delta$. We let $\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^1_\Sigma)$ be, respectively, the continuous affine functions on $\mathcal{H}^1$ and $\mathcal{H}^1_\Sigma$, and we let $\mathcal{A}^+(\mathcal{H}^1)$ and $\mathcal{A}^+(\mathcal{H}^1_\Sigma)$ denote the corresponding sets of nonnegative elements. Furthermore, let Δ_e^Σ denote the set of extreme points of the compact convex set $\mathcal{H}^1_\Sigma$.*

We shall discuss in this section the property that $H(\Delta)$ is a *simplicial function space*, or, briefly, *simplicial*. This means that for every compact set $K \subset \partial_e\Delta$ and any $f \in \mathcal{C}(K)$ there exists an $h \in H(\Delta)$ such that $h|K = f$ and $\|h\|_\Delta = \|f\|_K$ (in other words, the *weak Dirichlet problem* is solvable). We shall show here that the simpliciality of $H(\Delta)$ is equivalent to the property that the set of r-sturdy functions in $\mathcal{H}^1$ forms a Choquet simplex.

Note that

$$\mathcal{A}\left(\mathcal{H}^1\right)|\mathcal{H}^1_\Sigma \subset \mathcal{A}\left(\mathcal{H}^1_\Sigma\right).$$

In Proposition 5.4, we showed that the mapping $T : y \longmapsto Q(\cdot, y)$ from $\widehat{X}$ into $\mathcal{H}^1$ is continuous, and that $T|\Delta$ is a homeomorphism from Δ into $\mathcal{H}^1$. Also recall that for each $f \in \mathcal{C}(\Delta)$, H_f denotes the element of $\mathcal{H}$ that extends f to X.

9.2. **Lemma.** *For any point $x \in X$, and any $a \in \mathcal{A}(\mathcal{H}^1_\Sigma)$,*

$$H_{a\circ T}(x) = \int_\Delta a \circ T(z)\, \mu_x(dz) = \int_\Delta a\left(Q(\cdot, z)\right)\, \mu_x(dz) = a \circ T(x).$$

Proof. Since μ_x is a probability measure on Δ, it suffices to prove the result for affine functions of the form E_y where y is an arbitrary point of X and E_y denotes evaluation at y. Now

$$\int_\Delta E_y \circ T(z)\, \mu_x(dz) = \int_\Delta Q(y, z)\, \mu_x(dz) = Q(y, x) = E_y \circ T(x). \quad \square$$

9.3. **Proposition.**

(1) *The set $\mathcal{A}^+(\mathcal{H}^1_\Sigma) \circ T = H^+(\widehat{X})$.*
(2) *If $\mathcal{H}^1_\Sigma$ is a face, then $\mathcal{A}^+(\mathcal{H}^1) \circ T = H^+(\widehat{X})$.*

Proof.

(1) If $a \in \mathcal{A}^+(\mathcal{H}^1_\Sigma)$, then $a \circ T$ is a nonnegative, continuous function on $\widehat{X}$. It follows from Lemma 9.2 that on X, this function is equal to the extension in $\mathcal{H}$ of its values on Δ. Hence $a \circ T$ is in $H^+(\widehat{X})$. Now fix $h \in H^+(\widehat{X})$. For each $g \in \mathcal{H}^1_b$, let F_g be the corresponding L^∞ function on Δ. Set $a(g) = \int_\Delta h(z)F_g(z)\sigma(dz)$. This is affine, nonnegative, and continuous on the bounded, nonnegative harmonic functions with the ucc-topology, and we may extend it to an element of $\mathcal{A}^+(\mathcal{H}^1_\Sigma)$. The extension represents an element $\widetilde{h} \in H^+(\widehat{X})$, and for each $g \in \mathcal{H}^1_b$,

$$a(g) = \int_\Delta h(z)F_g(z)\sigma(dz) = \int_\Delta \widetilde{h}(z)F_g(z)\sigma(dz).$$

Since h and $\widetilde{h}$ are continuous on Δ, it follows that $h = \widetilde{h}$ on $\widehat{X}$.

(2) Assume $\mathcal{H}^1_\Sigma$ is a face, and fix $h \in H^+(\widehat{X})$. By Part (1), $h = a \circ T$ for some $a \in \mathcal{A}^+(\mathcal{H}^1_\Sigma)$. We may extend a to an element of $\mathcal{A}^+(\mathcal{H}^1)$ since $\mathcal{H}^1_\Sigma$ is a closed face of a Choquet simplex. □

9.4. **Corollary.** *The set* $\Delta_e^\Sigma = \partial_e \Delta \subset \Delta_{\text{reg}}$.

Proof. The first equality follows from Proposition 9.3. For the inclusion, fix $z_0 \in \partial_e \Delta$, and let $\mathcal{F}$ be an ultrafilter on X such that $\lim \mathcal{F} = z_0$. If $\mu := \lim_{x,\mathcal{F}} \mu_x$, then for every $h \in H(\Delta)$

$$\int h\, d\mu = \lim_{x,\mathcal{F}} \int h(z) Q(x,z)\, \sigma(dz) = \lim_{\mathcal{F}} H_h = h(z_0),$$

hence $\mu = \varepsilon_{z_0}$, and therefore $z_0 \in \Delta_{\text{reg}}$. □

9.5. **Corollary.** *The function space* $H(\Delta)$ *is simplicial if and only if* $\mathcal{H}^1_\Sigma$ *is a Choquet simplex.*

Proof. This follows from Proposition 9.3 by noting that $\mathcal{H}^1_\Sigma$ is a Choquet simplex if and only if $\mathcal{A}(\mathcal{H}^1_\Sigma)$ is a simplicial function space. □

We now can give further conditions for $\mathcal{H}^1_\Sigma$ to be a face.

9.6. **Theorem.** *The following assertions are equivalent:*

(1) $\mathcal{H}^1_\Sigma$ *is a face of* $\mathcal{H}^1$.
(2) $\mathcal{A}^+(\mathcal{H}^1) \circ T = H^+(\widehat{X})$.
(3) $\Delta_e = \partial_e \Delta$.

Proof. (1) ⇒ (2): This follows from Lemma 9.2.

(2) ⇒ (3): Using Proposition 9.3, we see that $\partial_e \Delta = \mathcal{E}^1 \cap \Delta = \Delta_e$.

(3) ⇒ (1): Fix $z_0 \in \Delta \setminus \Delta_e = \Delta \setminus \partial_e \Delta$. There exists a probability measure μ on Δ such that $\mu \neq \varepsilon_{z_0}$, $\mu(\Delta_e) = \mu(\partial_e \Delta) = 1$, and

$$\int h\, d\mu = h(z_0) \text{ for all } h \in H(\Delta).$$

It follows from Lemma 9.2 and Proposition 9.3 that

$$Q(\cdot, z_0) = \int Q(\cdot, z) \mu(dz),$$

whence (1) follows from Proposition 8.7. □

9.7. **Remark.** If $\mathcal{H}^1_\Sigma$ is a face, then it is also a Choquet simplex. Hence, in this case, $H(\Delta)$ is a simplicial function space by Corollary 9.5. Note that Example 10.2, (3), describes a harmonic system for which $\mathcal{H}^1_\Sigma$ is not a face; it is nevertheless a Choquet simplex, i.e., $H(\Delta)$ is still a simplicial function space. We note that in [14], T. Ikegami has studied resolutive compactifications of harmonic spaces where the spaces corresponding to $H(\Delta)$ are simplicial.

10. $\mathbf{h_0}$-Sturdy Functions

Let $(X, \mathcal{H}, r)$ be a harmonic system, and let σ_0 be a probability measure on $\mathcal{H}^1$ supported by $\mathcal{E}^1$. Fix $h_0 > 0$ such that for all $x \in X$,

$$h_0(x) := \int_{\mathcal{E}^1} g(x)\, \sigma_0(dg).$$

10.1. **Definition.** *A function $h \in \mathcal{H}^+$ will be called h_0-**sturdy** if there exists a sequence $\langle h_n \rangle$ in $\mathcal{H}^+$ such that*

(1) h_n/h_0 *is bounded for all* $n \in \mathbb{N}$,
(2) $\langle h_n \rangle$ *converges u.c.c. to* h.

By Remark 8.1 the set of all 1-sturdy harmonic functions is the set of all sturdy functions. Using the notation of Example 3.3, and applying the foregoing results to the harmonic system $(X, \widetilde{\mathcal{H}}, \widetilde{r})$, we have the following consequences:
Let $\Sigma_0 :=$ support(σ_0). Then $h \in \mathcal{H}^+$ is h_0-bounded if and only if there exists a measure μ on Σ_0 such that for all $x \in X$,

$$\frac{h(x)}{h_0(x)} = \int \widetilde{Q}(x, z)\mu(dz).$$

We call the corresponding compactification $\widetilde{X}$ such that $\widetilde{X} = X \cup \widetilde{\Delta}$ the h_0-*compactification* of X (with respect to $(X, \mathcal{H}, r)$).
If for all $x, y \in X$

$$Q_0(x, y) := \frac{1}{h_0(y)} \int_{\mathcal{E}^1} g(x) g(y)\, \sigma_0(dg),$$

then

$$\widetilde{Q}(x, y) = \frac{Q_0(x, y)}{h_0(x)}.$$

Hence $h \in \mathcal{H}^+$ is h_0-bounded if for some measure μ on Σ_0

$$h(x) = \int Q_0(x, z)\mu(dz).$$

10.2. **Example.**

(1) Let $h_0 \in \mathcal{E}^1$. The corresponding h_0-compactification is just the one-point-compactification, since for all $x, y \in X$

$$\widetilde{Q}(x, y) = \frac{Q_0(x, y)}{h_0(x)} = 1,$$

and $\{\lambda h_0 : \lambda \geq 0\}$ is the full set of h_0-bounded harmonic functions.

(2) Choose a probability measure σ_0 on $\mathcal{H}^1$ such that

$$\sigma_0(\mathcal{H}^1 \setminus \mathcal{E}^1) = 0 \text{ and } \mathcal{E}^1 \subset \text{support}(\sigma_0).$$

If for all $x \in X$ we set

$$h_0(x) := \int g(x)g(y)\,\sigma_0(dg),$$

then the h_0-compactification gives a concrete integral representation for all $h \in \mathcal{H}^1$. That is, every $h \in \mathcal{H}^1$ is h_0-bounded, and therefore

$$h(x) = \int Q_0(x,z)\,\mu_h(dz) \text{ for all } x \in X.$$

(3) The following counterexample, given to us by R. Wittmann, shows that the set of r-sturdy harmonic functions is in general not a face.
Let $(X, \mathcal{H}, r)$ be the harmonic system formed by the ordinary harmonic functions on an open, connected domain $X \subset \mathbb{R}^n$ where $x_0 \in X$, and $r = \varepsilon_{x_0}$. Assume that $\mathcal{E}^1$ is not closed, and choose $g_0 \in \overline{\mathcal{E}^1} \setminus \mathcal{E}$ with corresponding minimal representing measure μ_0. Let $\langle h_n \rangle$ be a sequence in $\mathcal{E}^1$ such that

(a) $\langle h_n \rangle$ converges u.c.c. to g_0, and
(b) $\mu_0(\mathcal{E}^1 \setminus \{h_n : n \in \mathbb{N}\}) > 0.$

Furthermore, let $\sigma_0 := \sum_{n=1}^{\infty} \frac{1}{2^n}\varepsilon_{h_n}$,

$$\Sigma_0 := \text{support}(\sigma_0) = \{h_n : n \in \mathbb{N}\} \cup \{g_0\},$$

and

$$h_0(x) := \int g(x)\,\sigma_0(dg) \text{ for all } x \in X.$$

By Remark 3.3, $(X, \widetilde{\mathcal{H}}, \widetilde{r})$ is a harmonic system where

$$\widetilde{\mathcal{H}} = \left\{\frac{h}{h_0} : h \in \mathcal{H}\right\}$$
$$\widetilde{r}\left(\frac{h}{h_0}\right) := \frac{r(h)}{r(h_0)} \text{ for all } h \in \mathcal{H}.$$

The canonical map $\varphi : \mathcal{H} \longrightarrow \widetilde{\mathcal{H}}$, defined by $\varphi(h) = \frac{h}{h_0}$, $h \in \mathcal{H}$, defines a homeomorphism of $\overline{\Delta}_e$ onto $\overline{\widetilde{\Delta}}_e$. Therefore $\widetilde{\sigma}_0 := \varphi(\sigma_0)$ is the representing measure of 1 in $\widetilde{\mathcal{H}}^1$. Since $\widetilde{g}_0 = \varphi(g_0) \in \widetilde{\Sigma}_0 := \varphi(\Sigma_0)$, it follows that $\widetilde{g}_0$ is an $\widetilde{r}$-sturdy harmonic function.

Let

$$\alpha_1 = \int_{\complement\Sigma_0} g(x_0)\mu_0(dg), \quad \alpha_2 := \int_{\Sigma_o} g(x_0)\mu_0(dg) = 1 - \alpha_1.$$

Then

$$\widetilde{g}_0 = \alpha_1 \widetilde{h}_1 + \alpha_2 \widetilde{h}_2 \text{ with } 0 < \alpha_1 < 1,$$

where

$$\widetilde{h}_1 = \frac{1}{h_0} \int_{\complement\Sigma_0} g\,\mu_0(dg), \quad \widetilde{h}_2 := \frac{1}{h_0} \int_{\Sigma_0} g\,\mu_0(dg).$$

On the other hand, $\widetilde{h}_1$ is not $\widetilde{r}$-sturdy, so $\widetilde{\mathcal{H}}^1_\Sigma$ is not a face of $\widetilde{\mathcal{H}}^1$.

Note that in this example, $\widetilde{\mathcal{H}}^1_\Sigma$ is even a Bauer simplex, i.e., a Choquet simplex with the closed set $\widetilde{\Sigma}_0$ as set of extreme points.

11. Boundary Behavior of Sturdy Functions

Since Σ is a compact metric space, the family $\mathcal{B}$ of Borel sets in Σ is countably generated. It follows from the result of H. von Weizsäcker and the second author given in the next section that with respect to our measure σ, there is "zero-set mapping" $\nu \mapsto Z_\nu$ from the family M of finite Borel measures defined on Σ into $\mathcal{B}$. As in [6], the mapping Z determines an "optimal Fatou approach filter system" $z \mapsto \mathcal{F}_z$ at the points z of Σ. This filter system produces limits at points of Σ for the extensions h_ν in $\mathcal{H}$ of measures ν in M. For each $z \in \Sigma$, a filter base $\widetilde{\mathcal{F}}_z$ for $\mathcal{F}_z$ is given by sublevel sets in X formed using r-sturdy functions. To be exact, since $h_\sigma = 1$,

$$\widetilde{\mathcal{F}}_z := \left\{ \{h_\nu < 1\} : \nu \in M,\ x \in Z_\nu \right\}.$$

For each $\mu \in M$, we will let $d\mu/d\sigma$ denote the Radon–Nikodým derivative of the absolutely continuous part of μ with respect to σ. An application of [6] now gives the following result.

11.1. **Theorem.** *For each $\mu \in M$, let $h_\mu \in \mathcal{H}^1_\Sigma$ be defined by setting*

$$h_\mu(x) = \int_\Sigma Q(x,z)\,\mu(dz)$$

for all $x \in X$. Then for σ-almost all $z \in \Sigma$, $\lim_{\mathcal{F}_z} h_\mu$ exists and

$$\lim_{\mathcal{F}_z} h_\mu = \frac{d\mu}{d\sigma}(z).$$

Moreover, for all z in the zero-set Z_μ,

$$\lim_{\mathcal{F}_z} h_\mu = 0.$$

The filters $\mathcal{F}_z$ are the coarsest ones with these properties.

11.2. **Corollary.** *If for μ and ν in M the function $h_\mu = h_\nu$, then*

$$\frac{d\mu}{d\sigma}(z) = \frac{d\nu}{d\sigma}(z)$$

for σ-almost all $z \in \Sigma$.

11.3. **Corollary.** *If $z_0 \in \Sigma \setminus \Delta_e$, then $Q(\cdot, z_0)$ is neither bounded nor even quasibounded.*

Proof. The function $Q(\cdot, z_0)$ is represented by unit mass at z_0, which is singular with respect to σ. Therefore, $Q(\cdot, z_0)$ cannot be given by a density with respect to σ on Σ (see Proposition 8.5). □

11.4. **Corollary.** *Suppose $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$. Then for each $h \in \mathcal{H}^1_\Sigma$, the minimal representing measure μ_h satisfies the equation*

$$\lim_{\mathcal{F}_z} h = \frac{d\mu_h}{d\sigma}(z)$$

for σ-almost every $z \in \Sigma$.

11.5. **Corollary.** *Suppose $\mathcal{H}^1_\Sigma$ is a face of $\mathcal{H}^1$. Then for each point $z_0 \in \Sigma \setminus \Delta_e$, the minimal representing measure for $Q(\cdot, z_0)$ is singular with respect to σ.*

Proof. The function $Q(\cdot, z_0)$ is represented by unit mass at z_0, which is singular with respect to σ. By Corollary 11.2, the minimal representing measure for $Q(\cdot, z_0)$ is also singular with respect to σ. □

12. Appendix: The Existence of Zero-Set Mappings

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In [6], Bliedtner and Loeb showed that there is an optimal method that obtains boundary limits for functions in a harmonic space once one determines in a consistent way where zero limits should occur. The required zeros are given by a "zero-set mapping" $Z : \mu \mapsto Z_\mu$ from the set of finite measures M on the boundary Y of the appropriate harmonic space into the collection of subsets of Y. To make that boundary limit result available for this paper, we show in this appendix that there always exists a zero-set mapping with the properties required in [6] when the underlying measure space is countably generated. Our mapping will, in fact, have an additional property that keeps zero sets small for singular measures, and thus produces coarse boundary approach filters. The result we will establish here also provides a zero-set mapping that can be used for measure differentiation as discussed by Bliedtner and Loeb in [7]. This broad applicability requires a rather general setting.

We work with a set Y and a σ-algebra $\mathcal{M}$ in Y. We let M be the collection of all finite measures on the measurable space $(Y, \mathcal{M})$, and we fix a nonzero "reference measure" σ in M. If $A \subset Y$ is contained in an $\mathcal{M}$-measurable set of σ measure zero, we may write $\sigma(A) = 0$.

12.1**. Definition.** *A function Z mapping M into the subsets of Y is a* ***zero-set mapping*** *if the following five conditions hold for all ν, $\mu \in M$, all measurable sets $E \subseteq Y$, and all $c > 0$ in $\mathbb{R}$:*

$$\text{i) } Z_\nu \cap Z_\mu \subseteq Z_{\nu+\mu}, \ \text{ii) } Z_{c\nu} = Z_\nu, \ \text{iii) } Z_\nu = \emptyset \ \textit{if } \sigma \le \nu \textit{ on } Y,$$
$$\text{iv) } Z_0 = Y, \quad \text{v) } \textit{if } \nu(E) = 0, \textit{ then } \sigma(E \setminus Z_\nu) = 0.$$

Given $\nu \in M$, we call Z_ν the ***zero set*** *for ν. A zero-set mapping is called* ***positive*** *if for all ν, $\mu \in M$, $Z_\nu \cap Z_\mu = Z_{\nu+\mu}$; that is, $\nu \le \mu \Rightarrow Z_\mu \subseteq Z_\nu$. A zero-set mapping is called* ***proper for*** *$\mu \in M$ if there is a set $A_\mu \supseteq Z_\mu$ with $A_\mu \in \mathcal{M}$ and $\mu(A_\mu) = 0$; it is called M-**proper** if it is proper for each element of M. A zero-set mapping is called* ***measurable*** *if each zero set is itself in $\mathcal{M}$.*

The equivalence of the conditions for positivity follows from the facts that for positive measures, $\nu \leq \nu + \mu$, $\mu \leq \nu + \mu$, and if $\nu \leq \mu$ then $\mu = \nu + (\mu - \nu)$. Zero set mappings are used in [6] and [7] to obtain Radon–Nikodým derivatives for the σ-absolutely continuous parts of the measures in M; these derivatives must vanish at the points of their corresponding zero-sets. It follows that for measures absolutely continuous with respect to σ, the zero-set mapping will be proper. It is not necessary, however, for the applications in [6] and [7] that the zero-set mapping be proper for measures singular with respect to σ. On the other hand, coarser filters, and therefore better limits, are produced when this is the case. We note also that a zero-set mapping retains its properties when restricted to appropriate subclasses of M such as the set of all measures $f\sigma$, where $f \in L^p(\sigma)$, $1 \leq p \leq \infty$.

12.2. **Theorem.** *Assume $\mathcal{M}$ is countably generated. Then there exists a positive, M-proper, measurable zero-set mapping $Z : M \to \mathcal{M}$.*

Proof. Let $\langle \pi_n : n \in \mathbb{N} \rangle$ be an increasing sequence of finite measurable partitions of Y such that $\mathcal{M}$ is the smallest σ-algebra containing all of the sets in all of the partitions π_n. Let

$$Y_\sigma = Y \setminus \bigcup_{n=1}^{\infty} \left(\bigcup \{A \in \pi_n : \sigma(A) = 0\} \right).$$

That is, we remove from Y all σ-null partition sets that occur in any partition. Without loss of generality, we may now assume that $\pi_1 = \{Y_\sigma, Y \setminus Y_\sigma\}$, and we work with Y_σ. The zero-set $Z_0 = Y$. All other zero-sets will be subsets of Y_σ. Clearly, all partition sets in Y_σ have positive σ measure.

Given $n \in \mathbb{N}$, we let $\mathcal{M}_n$ denote the algebra generated by the sets in the partition π_n, and for each $\mu \in M$, we set

$$f_n^\mu = \sum_{\substack{A \in \pi_n \\ A \subseteq Y_\sigma}} \frac{\mu(A)}{\sigma(A)} \cdot \chi_A.$$

Here, χ_A denotes the characteristic function of A on Y_σ. For each nonzero $\mu \in M$, we set

$$Z_\mu = \left\{ y \in Y_\sigma : \lim_{n \to \infty} f_n^\mu(y) = 0 \right\}.$$

It is easy to see that the mapping $\mu \to Z_\mu$ is positive and measurable, and that it satisfies Conditions i, ii, and iii. Condition iv has been satisfied by setting $Z_0 = Y$.

To verify Condition v, we fix a set $E \in \mathcal{M}$ with $E \subseteq Y_\sigma$, and we fix a measure $\nu \in M$ with $\nu(E) = 0$. We must show that $\sigma(E \setminus Z_\nu) = 0$.

However, since for any positive integer k, $Z_\nu = Z_{k\nu}$, we only need to show that $\limsup f_n^\nu \leq 1$ for σ-almost all points of E. We will write f_n instead of f_n^ν to simplify our notation. As in the proof of the martingale convergence theorem in Section 3 of [5], for each $A \in \mathcal{M}$ and each $k \in \mathbb{N}$, we set $A^k := \left\{ y \in A : \sup_{n \geq k} f_n(y) > 1 \right\}$. By the maximal inequality, if $A \in \mathcal{M}_k$, $\nu(A^k) \geq \sigma(A^k)$. Recall that this follows by setting $B^{k-1} := \emptyset$, $B^n := \{y \in A : f_n(y) > 1\} \setminus \cup_{i=k-1}^{n-1} B^i$ for $n \geq k$, and noting that

$$\nu\left(A^k\right) = \sum_{i=k}^{\infty} \nu(B^i) = \sum_{i=k}^{\infty} \int_{B^i} f_i d\sigma \geq \sum_{i=k}^{\infty} \sigma\left(B^i\right) = \sigma\left(A^k\right).$$

We will finish the proof of Condition v by showing that $\sigma\left(\cap_{k=1}^{\infty} E^k\right) = 0$. Fix $\varepsilon > 0$. Since $\mathcal{M}$ is generated by the $\mathcal{M}_n$'s, there is $k \in \mathbb{N}$ and a set $A \in \mathcal{M}_k$ such that the symmetric difference with E satisfies $(\sigma + \nu)(A \Delta E) < \varepsilon$. Using the maximal inequality, we have

$$\begin{aligned} \sigma\left(E^k\right) &\leq \sigma(E \setminus A) + \sigma\left(A^k\right) \\ &\leq \sigma(E \setminus A) + \nu\left(A^k\right) \leq \sigma(E \setminus A) + \nu(A \setminus E) < \varepsilon, \end{aligned}$$

and Condition v follows.

Finally, we show that Z is an M-proper zero-set mapping. Fix $\mu \in M$. For each $n \in \mathbb{N}$, set

$$g_n := \frac{f_n^\mu}{1 + f_n^\mu}.$$

Then g_n is the Radon–Nikodým derivative on Y_σ of $\mu|\mathcal{M}_n$ with respect to $(\mu + \sigma)|\mathcal{M}_n$. It follows from the Martingale Convergence Theorem applied to the positive martingale $\langle f_n^\mu : n \in \mathbb{N} \rangle$ that the g_n's converge σ-a.e. on Y_σ to a positive function g bounded by 1. If $A \in \mathcal{M}_n$ for some $n \in \mathbb{N}$, we have for each $k \geq n$,

$$\mu(A) = \int_A g_k \, d(\mu + \sigma) = \int_A g \, d(\mu + \sigma).$$

Fix $B \in \mathcal{M}$ and $\varepsilon > 0$. Again, there is $k \in \mathbb{N}$ and a set $A \in \mathcal{M}_k$ with $(\sigma + \mu)(A \Delta B) < \varepsilon/2$. We now have

$$\begin{aligned} &\left| \mu(B) - \int_B g d(\mu + \sigma) \right| \\ &\leq |\mu(B) - \mu(A)| + \left| \int_A g d(\mu + \sigma) - \int_B g d(\mu + \sigma) \right| < \varepsilon. \end{aligned}$$

It follows that $g = d\mu/d(\mu + \sigma)$. Since $g \equiv 0$ on Z_μ, $\mu(Z_\mu) = 0$. $\square$

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