NONSTANDARD INTEGRATION THEORY IN TOPOLOGICAL VECTOR LATTICES

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Abstract. This paper develops a Daniell-Stone integration theory in topological vector lattices. Starting with an internal, vector valued, positive linear functional I on an internal lattice of vector valued functions, we produce a nonstandard hull valued integral J satisfying the monotone convergence theorem. Nonstandard hulls form a natural extension of infinite dimensional spaces and are equivalent to Banach space ultrapower constructions. The first application of our integral is a construction of Banach limits for bounded, vector valued sequences. The second example yields an integral representation for bounded and quasibounded harmonic functions similar to that of the Martin boundary. The third application uses our general integral to extend the Bochner integral.

1. **Introduction**

Nonstandard Analysis has produced a large body of results in real valued measure and probability theory. Starting with the first author's paper [12] and Anderson's application to Brownian motion and the Itô Integral [2], many areas of mathematics, including mathematical economics and mathematical physics, have benefited from the application of these techniques (see, for example, [1]). In [15], a Daniell-Stone approach to the theory was developed by the first author and augmented with results for an integral formed in a nonstandard hull of a Banach Lattice. Such results have broad implications since nonstandard hulls form a natural extension of infinite dimensional spaces and are equivalent to Banach space ultrapower constructions. Shortly after the completion of [15], work on this article began; the authors

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set out to combine the theory of nonstandard hulls and that of special measure spaces, now called “Loeb spaces” in the literature. Loeb spaces are standard measure spaces with a complete, real valued measure formed in a nonstandard model. As demonstrated by the work of Keisler (e.g.,[9]) and the recent work of Sun (e.g., [26]), these spaces have special closure properties not shared by even Lebesgue measure spaces. To illustrate this point, we present at the end of this section a Lyapunov theorem for measures on a Loeb space taking values in the nonstandard hull of $\mathbb{R}^n$ where n is infinite. Preliminary results of the authors' research for this article were discussed in [16], and updated preprints have been used by several colleagues in later papers, now in the literature, on nonstandard analysis and vector valued integrals.

Our concern in this paper is a Daniell-Stone integration theory in topological vector lattices. In particular, we establish a monotone convergence theorem for the corresponding integrals. We work with an internal, vector valued, positive linear functional I and a corresponding standard integral J with values in a nonstandard hull of an internal vector lattice. Our theory is quite general. This is due in part to the lack of continuity assumption on the initial integral I. Such an assumption is usual in the Daniell approach to the integral (see, for example, Royden [24]), and is used by Kluvanec [10] in his work on the Daniell vector integral. Since vector measures are bounded, we do assume finiteness for I, and therefore J, when applied to finite valued functions. Another source of generality in our theory is the use of a weak notion of solidness for the neighborhoods of 0 in our spaces. This allows us to treat harmonic functions with the topology of uniform convergence on compact sets as an example of the theory.

Nonstandard hulls were introduced by Luxemburg [18] and developed further by Henson and Moore (e.g., see [7]). The standard theory of ordered topological vector spaces can be found in Luxemburg and Zaanen [19] and in Peressini [22]. As noted, preliminary results from this paper were described in [16]. Sun [25] has now developed a different approach to integration theory in nonstandard hulls of Banach spaces. An approach to nonstandard vector integration using the weak topology can be found in [20] and [21].

After establishing a monotone convergence theorem for our general integral, we consider several applications. The first is a construction of Banach limits for bounded, vector valued sequences which generalizes Robinson's [23] construction of Banach limits for bounded, real valued sequences. Here, the Banach limit is expressed in terms of our general integral J. Even when one starts with a standard sequence of elements in a standard space such as ℓ^1, the Banach limit given by J may only exist in the nonstandard hull of the original space.

Our second example starts with a regular boundary X in the monad of infinity for a harmonic space as discussed in [13]. An example of such a boundary is the circle of radius $1-\varepsilon$ where ε is a positive infinitesimal and the harmonic space is formed by functions that are harmonic in the usual sense defined on the unit disk. The internal integral $I(f)$ of an internal continuous function f defined on X is the solution h_f of the internal Dirichlet problem. The range of I is a subset of the space of internal harmonic functions with the internal topology given by uniform convergence on compact sets. Using a locally solid topology for harmonic functions, we map the range of the standard integral J onto the set of bounded and quasibounded harmonic functions defined on the full standard domain of the harmonic space. Thus from J we obtain an integral representation for bounded and quasibounded harmonic functions similar to that of the Martin boundary and the boundary developed by the first author in [13] and [14].

In the third (last) example, we will use our general integral to characterize and extend the Bochner Integral. For appropriate functions on a Loeb space taking their values in the nonstandard hull $\hat{\mathbf{E}}$ of a Banach lattice $\mathbf{E}$, we will obtain the $\hat{\mathbf{E}}$-valued Bochner Integral. We will also obtain a natural extension of that integral to a class of nonmeasurable functions when the dimension of $\mathbf{E}$ is infinite. If $\mathbf{E}$ is complete, our integral for $\mathbf{E}$-valued functions will be the usual Bochner Integral.

The applications given in this article have been developed to illustrate our general theory. As B. Zimmer has already shown in [28], the results on the Bochner Integral can be extended to arbitrary Banach spaces where no lattice structure may exist. Preliminary work of the authors shows that our general theory can also be used to formulate a representation of the stochastic integral. This topic will be explored in a later article.

Throughout this paper, we work with an $\aleph$-saturated nonstandard model, where $\aleph$ is an uncountable cardinal number. We denote the set of real numbers by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. We adopt the convention of internal set theory by calling an element $\rho \in {}^*\mathbb{R}$ **limited** when $|\rho| \leq n$ for some $n \in \mathbb{N}$ and **unlimited** when $|\rho| > n$ for all $n \in \mathbb{N}$. The unlimited elements of ${}^*\mathbb{N}$ are denoted by ${}^*\mathbb{N}_\infty$. When ρ is limited in ${}^*\mathbb{R}$, we write $st(\rho)$ or ${}^\circ\rho$ to denote the standard part of ρ. That is, ${}^\circ\rho$ is the unique standard real number infinitely close to ρ (i.e., ${}^\circ\rho \approx \rho$). By a partial ordering on a set S, we mean a transitive relation $\leq$ such that we have both $a \leq b$ and $b \leq a$ in S if and only if $a = b$. For notions and notations not defined here and for the foundations of nonstandard analysis, we refer the reader to the books of Hurd and Loeb [8] and Albeverio, Fenstad, Hoegh-Krøhn, and Lindstrøm [1]. A lattice approach to nonstandard scalar valued measure theory can be found in [8] and [15].

Before proceeding, we illustrate the closure properties of Loeb measure spaces when combined with nonstandard hulls by establishing the following Lyapunov theorem. Here, we use a nonstandard hull formed from an internal space rather than the extension of a standard space.

Theorem 1.1. *Fix a hyperfinite set A, an element $n \in {}^*\mathbb{N}_\infty$, and an internal norm $\|\cdot\|$ on ${}^*\mathbb{R}^n$. For each $a \in A$, let $\mathbf{v}(a)$ be a vector in ${}^*\mathbb{R}^n$, and let $u(a)$ be a positive number in ${}^*\mathbb{R}$ with $n \cdot u(a) \approx 0$ and $\|\mathbf{v}(a)\| \leq u(a)$. Assume that $\sum_{a\in A} u(a)$ is limited in ${}^*\mathbb{R}$. Let μ and $\mathbf{V}$ be the internal measures formed by summing u and $\mathbf{v}$ respectively over internal subsets of A, and let $(A, \mathcal{M}, \widehat{\mu})$ be the Loeb space formed from u. Let $\widehat{\mathbf{V}}$ be the external, σ-additive, nonstandard hull valued measure on $(A, \mathcal{M})$ formed as in [20] using $\widehat{\mu}$ as a control measure. Then the range of $\widehat{\mathbf{V}}$ is both convex and compact.*

Proof. It is shown in [11] that the range of the internal measure $\mathbf{V}$ is almost convex. That is, given internal sets $B \subseteq A$ and $C \subseteq A$ and a number $\lambda \in {}^*\mathbb{R}$ with $0 < \lambda < 1$, there is an internal set $D \subseteq A$ with $\|\mathbf{V}(D) - [\lambda\mathbf{V}(B) + (1-\lambda)\mathbf{V}(C)]\| \approx 0$. The convexity of the range of $\widehat{\mathbf{V}}$ follows. By saturation, the limit of a Cauchy sequence in the range of $\widehat{\mathbf{V}}$ is also in the range, whence the range is closed. □

Part 1. The General Theory

2. Nonstandard Hulls in Internal Topological Vector Lattices

Recall that a vector space $\mathbf{B}$ over $\mathbb{R}$ equipped with a neighborhood base $\mathcal{U}$ of 0 and a partial ordering $\leq$ is called a **topological vector lattice** if the following conditions hold:

- **(A):** The topology on $\mathbf{B}$ given by $\mathcal{U}$ makes $\mathbf{B}$ into a Hausdorff topological vector space over $\mathbb{R}$.
- **(B):** For all a, b, $c \in \mathbf{B}$ and all $\lambda \in \mathbb{R}$, $a \leq b$ implies $a + c \leq b + c$ and $a \leq b$ implies $\lambda \cdot a \leq \lambda \cdot b$ whenever $0 \leq \lambda$.
- **(C):** For all a, $b \in \mathbf{B}$, the least upper bound of a and b, denoted by $a \vee b$, exists in $\mathbf{B}$, and the greatest lower bound of a and b, denoted by $a \wedge b$, exists in $\mathbf{B}$.

We will assume an additional condition holds which is weaker than the usual assumption that the origin has a base of solid sets. Recall that the binary operations sup and inf, along with the corresponding unary operations $+$, $-$, and $|\cdot|$ are defined on $\mathbf{B}$ by $a^+ := a \vee 0$, $a^- := (-a) \vee 0$, $|a| := a \vee -a = a^+ + a^-$. These are called the **lattice operations** in $\mathbf{B}$, and $|a|$ is called the **lattice absolute value** of a.

A subset A of $\mathbf{B}$ is called **solid** if for each $a \in A$,

$$\{b \in \mathbf{B} : |b| \leq |a|\} \subseteq A.$$

We will call a subset A of $\mathbf{B}$ **semi-solid** if the following conditions hold:

(1): A is **absolute-order-convex** (see [27]). That is, for each $a \geq 0$ in A,

$$[-a, a] := \{b \in \mathbf{B} : -a \leq b \leq a\} = \{b \in \mathbf{B} : |b| \leq a\} \subseteq A.$$

(2): A is **circled.** That is, for each $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$, $\lambda \cdot A \subseteq A$.

Clearly, every solid set is semi-solid. Conversely, if A is semi-solid and $|a| \in A$ for each $a \in A$, then A is solid. In what follows, we will assume that our initial spaces satisfy the following assumption:

(D): The origin 0 in $\mathbf{B}$ has a neighborhood base consisting of semi-solid sets.

It is usual to assume that the origin has a base of solid sets since lattice operations are continuous when the origin has such a base. Recall that the real L^p spaces with the usual topology and pointwise ordering have solid neighborhoods of 0.

An example where the neighborhoods of 0 are semi-solid but not solid is provided by the space of differences of positive harmonic functions on the open unit disk D. The topology is that of uniform convergence on compact sets (see Section 6). An example of a normed space where the open balls centered at 0 are semi-solid but not solid is supplied by the linear functions $ax+b$ on the interval $[0, 1]$. For each such function, set

$$\|f\| = \max\{|f(1/4)|,\ |f(3/4)|\}.$$

If the initial topology on $\mathbf{B}$ is locally semi-solid but not locally solid, then one can replace it with a finer topology which is locally solid. To do so, one replaces each $U \in \mathcal{U}$ with the solid set

$$\widetilde{U} := \{a \in \mathbf{B} : |a| \in U\} \subseteq U.$$

If the non-locally solid topology is generated by a norm, then replacing the norm of each element with the norm of its lattice absolute value produces this finer topology. For the space consisting of differences of positive harmonic functions with the topology of uniform convergence on compact sets, this finer, locally solid topology can be obtained from an integral representation by using the total variation norm on the corresponding finite signed measures (see [17]). It is easy to see that for the general case, the space $\mathbf{B}$ is again a topological vector lattice when the topology is generated by the family $\{\widetilde{U} : U \in \mathcal{U}\}$.

For the development here, we will work with the original topology until we form a nonstandard hull. Our definitions of bounded and infinitesimal elements will be consistent, however, with the derived locally solid topology. We need to give some details in what follows about much of the structure we work with since, for our general theory, the nonstandard hull is formed from an internal topological vector lattice $(\mathbf{B}, +, \cdot, \mathcal{U}, \leq)$ which may not be the extension of a standard lattice. In this setting, the topology is an internal family of open sets closed under finite and therefore hyperfinite intersections and also closed with respect to unions of arbitrary internal collections. When we say that a set $A \subseteq \mathbf{B}$ is semi-solid we mean this in the internal sense. In particular, we assume that each $U \in \mathcal{U}$ is semi-solid in the internal sense.

A nonempty subset $\mathcal{U}_o$ of $\mathcal{U}$ of external cardinality less than the cardinality of saturation $\aleph$ will be called an **$\mathbb{R}$-neighborhood base of 0** if the following conditions hold for all sets U, $V \in \mathcal{U}_o$:

(1): There is a $W \in \mathcal{U}_o$ with $W \subseteq U \cap V$.
(2): There is a $W \in \mathcal{U}_o$ with $W + W \subseteq U$.

It follows from Condition 2 and the fact that each $W \in \mathcal{U}_o$ is circled that

(3): $\forall n \in \mathbb{N}$, $\exists W \in \mathcal{U}_o$ with $n \cdot W \subseteq U$, whence $\forall \alpha \in {}^*\mathbb{R}$ with $|\alpha| \leq n$, $\alpha \cdot W \subseteq U$.

Examples:

(I) Let $(\mathbf{B}, +, \cdot, \|\cdot\|, \leq)$ be an internal normed vector lattice. The internal topology on $\mathbf{B}$ comes from the internal norm $\|\cdot\|$. We assume that for each positive $\varepsilon \in {}^*\mathbb{R}$, the internal open ball $U_\varepsilon(0) := \{a \in \mathbf{B} : \|a\| < \varepsilon\}$ is semi-solid. Given a, $b \in \mathbf{B}$, if b is in every ball centered at 0 containing a, then $\|b\| \leq \|a\|$. Therefore, the balls $U_\varepsilon(0)$ will all be semi-solid iff for each a, $b \in \mathbf{B}$, $|b| \leq a \Rightarrow \|b\| \leq \|a\|$. These balls will all be solid iff for each a, $b \in \mathbf{B}$, $|b| \leq |a| \Rightarrow \|b\| \leq \|a\|$, i.e., iff the norm is a Riesz norm. In general, we set $\mathcal{U} = \{U_r(0) : r > 0,\ r \in {}^*\mathbb{R}\}$; for the external $\mathbb{R}$-neighborhood base of 0, we may choose the collection

$$\mathcal{U}_o = \{U_{1/n}(0) : n \in \mathbb{N}\}.$$

(II) Let $(\mathbf{E}, +, \cdot, \mathcal{W}, \leq)$ be a standard topological vector lattice. It is assumed that the sets in $\mathcal{W}$ are semi-solid. It is also assumed that the cardinality of saturation $\aleph$ has been chosen to be greater than the cardinality of $\mathcal{W}$. The collection $\mathcal{U}_o = \{{}^*W : W \in \mathcal{W}\}$ is an $\mathbb{R}$-neighborhood base of 0 for the internal topological vector lattice

$({}^*\mathbf{E}, {}^*+, {}^*\cdot, {}^*\mathcal{W}, {}^*\leq)$. By the continuity of scalar multiplication at 0, for each $a \in \mathbf{E}$ and each $W \in \mathcal{W}$, there is an $n \in \mathbb{N}$ with $|a| \in n \cdot W$, whence $|{}^*a| \in n \cdot {}^*W$ and ${}^*a \in n \cdot {}^*W$.

Returning to our general setting, we now fix an $\mathbb{R}$-neighborhood base $\mathcal{U}_o$ of 0 for $(\mathbf{B}, +, \cdot, \mathcal{U}, \leq)$. An element $a \in \mathbf{B}$ is called $\mathcal{U}_o$**-limited**, or just **limited**, if

$$\forall U \in \mathcal{U}_o, \ \exists n \in \mathbb{N} \ \text{ with } \ |a| \in n \cdot U.$$

An element $a \in \mathbf{B}$ is called $\mathcal{U}_o$**-infinitesimal**, or just **infinitesimal**, if

$$\forall U \in \mathcal{U}_o, \quad |a| \in U.$$

We will write $a \approx b$ when a and b are elements of $\mathbf{B}$ and $a - b$ is infinitesimal. In particular, $a \in \mathbf{B}$ is infinitesimal if and only if $a \approx 0$. We denote the $\mathcal{U}_o$-limited elements of $\mathbf{B}$ by $\mathcal{U}_o\text{-Lmd}\,\mathbf{B}$. If, as in Example II, $(\mathbf{E}, +, \cdot, \mathcal{W}, \leq)$ is a standard topological vector lattice and $\mathcal{U}_o = \{{}^*W : W \in \mathcal{W}\}$, we write $\text{Lmd}\,{}^*\mathbf{E}$ for $\mathcal{U}_o\text{-Lmd}\,{}^*\mathbf{E}$. In the following lemmas, we list some properties which follow easily from lattice manipulations and saturation.

Lemma 2.1. *The relation $\approx$ is an equivalence relation on $\mathbf{B}$. Moreover, for all a, $b \in \mathbf{B}$,*

(1) a is limited if a is infinitesimal;

(2) a is limited iff for all $H \in {}^*\mathbb{N}_\infty$, $(1/H) \cdot a$ is infinitesimal;

(3) a is infinitesimal iff there is an $H \in {}^*\mathbb{N}_\infty$ such that $H \cdot a$ is limited, and this is true iff there is an $H \in {}^*\mathbb{N}_\infty$ such that $H \cdot a$ is infinitesimal.

(4) If a and b are limited and λ is limited in ${}^*\mathbb{R}$, then $a + b$, λa, $a \vee b$, $a \wedge b$ and $|a|$ are all limited.

Although we do not assume solidness for elements of the internal neighborhood system $\mathcal{U}$, and therefore do not necessarily have internal continuity for the lattice operations, we do have the following result. The proof follows in part from Birkhoff inequalities such as $|(a \vee b) - (a' \vee b)| \leq |a - a'|$.

Lemma 2.2. *The operations $\vee$, $\wedge$, $|\cdot|$, $+$, and scalar multiplication by limited elements of ${}^*\mathbb{R}$ are $\mathcal{U}_o$-**continuous**. By this we mean that if λ and λ' are limited in ${}^*\mathbb{R}$ with $\lambda' \approx \lambda$ and if a, b, a', b' are in $\mathbf{B}$ with $a \approx a'$ and $b \approx b'$, then*

(1) $a + b \approx a' + b'$,

(2) $\lambda \cdot a \approx \lambda' \cdot a'$ if either $\lambda = \lambda'$ or a is $\mathcal{U}_o$-limited,

(3) $a \vee b \approx a' \vee b'$, $a \wedge b \approx a' \wedge b'$, and $|a| \approx |a'|$.

Lemma 2.3. *For any pair a, $b \in \mathbf{B}$, the following are equivalent:*
(1) $a \vee b \approx b$.
(2) $a \wedge b \approx a$.
(3) There exists an $a' \approx a$ *and* $b' \approx b$ *such that* $a' \leq b'$.
(4) For every $a' \approx a$, *there is a* $b' \approx b$ *with* $a' \leq b'$.
(5) For every $b' \approx b$, *there is an* $a' \approx a$ *with* $a' \leq b'$.

Given a, $b \in \mathbf{B}$, we will write $a \tilde{\leq} b$ if any one of the conditions of Lemma 2.3 holds. If $|a| \tilde{\leq} b$ and $b \approx 0$, then for some $b' \approx 0$, $|a| \leq b'$. Since each $U \in \mathcal{U}_o$ is semi-solid, $a \approx 0$.

For some applications, one may need to replace $\mathcal{U}_o$-Lmd $\mathbf{B}$ with a proper subset having similar properties. A nonempty subset $\mathbf{B}_o$ of $\mathbf{B}$ is called an **$\mathbb{R}$-sublattice of B** if for all a, $b \in \mathbf{B}_o$, all $c \in \mathbf{B}$, and all $\lambda \in \mathbb{R}$ we have the following:

(1): $a + b \in \mathbf{B}_o$,
(2): $\lambda \cdot a \in \mathbf{B}_o$,
(3): $a \vee b \in \mathbf{B}_o$ and $a \wedge b \in \mathbf{B}_o$,
(4): if $a \leq c \leq b$, then there is a $c' \approx c$ with $c' \in \mathbf{B}_o$.

It follows from Lemma 2.1 that $\mathcal{U}_o$-Lmd $\mathbf{B}$ is itself an $\mathbb{R}$-sublattice of $\mathbf{B}$. The standard elements of $\mathbb{R}$ form an $\mathbb{R}$-sublattice of ${}^*\mathbb{R}$. Suppose $\mathbf{B}_o$ is an $\mathbb{R}$-sublattice of $\mathbf{B}$ and for a, $b \in \mathbf{B}_o$ and $c \in \mathbf{B}$ we have only the weak inequality $a \tilde{\leq} c \tilde{\leq} b$. It still follows that for $c_0 := ((a \wedge b) \vee c) \wedge (a \vee b)$, we have $a \wedge b \leq c_0 \leq a \vee b$, and so for some element $c' \in \mathbf{B}_o$ we have $c' \approx c_0 \approx c$. In what follows, we will refer to Property (4) by saying that $\mathbf{B}_o$ is **nearly interval closed**.

Having fixed $\mathcal{U}_o$, we now also fix an $\mathbb{R}$-sublattice $\mathbf{B}_o$ of $\mathbf{B}$. The reader may, of course, make the simplifying assumption that $\mathbf{B}_o = \mathcal{U}_o$-Lmd $\mathbf{B}$. We will work with the following objects constructed from $\mathbf{B}$, $\mathbf{B}_o$, and $\mathcal{U}_o$:

$\bar{a} := \{b \in \mathbf{B} : b \approx a\}$ when $a \in \mathbf{B}$;
$\hat{\mathbf{B}}_o := \{\bar{a} : a \in \mathbf{B}, \bar{a} \cap \mathbf{B}_o \neq \emptyset\}$.
$\hat{V} := \{\bar{a} \in \hat{\mathbf{B}}_o : \overline{|a|} \subseteq V\}$ when $V \in \mathcal{U}_o$;
$\widehat{\mathcal{U}_o} := \{\hat{V} : V \in \mathcal{U}_o\}$.

For every pair $\bar{a}$, $\bar{b} \in \hat{\mathbf{B}}_o$ and $\lambda \in \mathbb{R}$, we set
$\bar{a} \hat{+} \bar{b} = \overline{a + b}$,
$\lambda \hat{\cdot} \bar{a} = \overline{\lambda \cdot a}$,
$\bar{a} \hat{\leq} \bar{b}$ if $a \tilde{\leq} b$.

Note that if $\mathbf{B}_o = \mathcal{U}_o$-Lmd $\mathbf{B}$, then $\hat{\mathbf{B}}_o := \{\bar{a} : a \in \mathcal{U}_o\text{-Lmd}\,\mathbf{B}\}$. Also note that the definition of $\hat{V}$ for $V \in \mathcal{U}_o$ is different from much of the literature, where each $V \in \mathcal{U}_o$ is mapped onto the set $\{\bar{a} \in \hat{\mathbf{B}}_o : \bar{a} \cap V \neq \emptyset\}$. As indicated above, our definition is consistent with the topology obtained by first replacing each internal neighborhood U with $\tilde{U} = \{a \in \mathbf{B} : |a| \in U\}$. Our definition is also consistent with earlier studies involving S-continuous seminorms p on spaces $\mathbf{B}$ where for each $b \in \mathbf{B}$, $p(b) = p(|b|)$. That is, if ε is positive in $\mathbb{R}$ and a is in $U = \{b \in \mathbf{B} : p(b) < \varepsilon\}$, then $a + i \in U$ for each $i \approx 0$ if and only if ${}^{\circ}p(a) < \varepsilon$.

By Lemma 2.2 and the closure properties of $\mathcal{U}_o$ and $\mathbf{B}_o$, the operations $\hat{+}$, and $\hat{\cdot}$ and the relation $\hat{\leq}$ are well defined and make $\hat{\mathbf{B}}_o$ into an ordered vector space. We now give further properties of this space. As above, for each $W \in \mathcal{U}_o$, $\tilde{W} := \{a \in \mathbf{B} : |a| \in W\}$; recall that $\tilde{W}$ is a solid subset of W with $\tilde{W} = W$ if W is solid.

Proposition 2.4. *Fix $a \in \mathbf{B}$ with $\bar{a} \cap \mathbf{B}_o \neq \emptyset$ and $U \in \mathcal{U}_o$. Then $\bar{a} \in \hat{U}$ iff $\overline{|a|} \subseteq \tilde{U}$ iff there is a $W \in \mathcal{U}_o$ with $|a| + \tilde{W} \subseteq \tilde{U}$; in this case $a + \tilde{W} \subseteq \tilde{U}$. Given sets $\hat{U}$, $\hat{V} \in \hat{\mathcal{U}}_o$ and $\lambda \in \mathbb{R}$, there is a $\hat{W} \in \hat{U}_o$ such that*

$$\hat{W} \subseteq \hat{U} \cap \hat{V}, \quad \hat{W} + \hat{W} \subseteq \hat{U}, \quad \text{and } \lambda \cdot \hat{W} \subseteq \hat{U}.$$

Proof. For the first part, if for any $i \approx 0$, $|a| + i \in U$, then for $j \approx 0$,

$$||a| + j| \leq |a| + |j| \in U.$$

Since U is semi-solid, $||a| + j| \in U$, whence $|a| + j \in \tilde{U}$. Since $\tilde{U} \subseteq U$, $\bar{a} \in \hat{U}$ iff $\overline{|a|} \subseteq \tilde{U}$. If there is a $W \in \mathcal{U}_o$ with $|a| + \tilde{W} \subseteq \tilde{U}$, then $\overline{|a|} \subseteq \tilde{U}$. Conversely, if for each $\tilde{W} \in \mathcal{U}_o$ there is a point $c \in (|a| + \tilde{W}) \backslash \tilde{U}$, then by saturation, there is a point c' in the set $\overline{|a|} \backslash \tilde{U}$, so $\bar{a} \notin \hat{U}$. If $|a| + \tilde{W} \subseteq \tilde{U}$, then for each $b \in \tilde{W}$, $a + b \in \tilde{U}$ since

$$|a + b| \leq (|a| + |b|) \in |a| + \tilde{W} \subseteq \tilde{U}.$$

Now fix $\hat{U}$, $\hat{V} \in \hat{\mathcal{U}}_o$, $\lambda \in \mathbb{R}$ and sets W_1, W_2, and W in $\mathcal{U}_o$ with

$$W_1 + W_1 \subseteq U, \quad \lambda \cdot W_2 \subseteq U, \quad \text{and} \quad W \subseteq W_1 \cap W_2 \cap V.$$

Then $\hat{W} \subseteq \hat{U} \cap \hat{V}$, and $\lambda \cdot \hat{W} \subseteq \hat{U}$. To see that $\hat{W} + \hat{W} \subseteq \hat{U}$, we fix $\bar{a}$ and $\bar{b}$ in $\hat{W}$ and note that for any $i \approx 0$, $|a + b| + i \in U$ since

$$||a + b| + i| \leq |a| + |b| + |i| \in W + W \subseteq U. \ \square$$

Theorem 2.5. *The space $(\hat{\mathbf{B}}_o, \hat{+}, \hat{\cdot}, \hat{\mathcal{U}}_o, \hat{\leq})$ is a topological vector lattice with*

$$\bar{a} \vee \bar{b} = \overline{a \vee b}, \quad \bar{a} \wedge \bar{b} = \overline{a \wedge b}, \quad \text{and } |\bar{a}| = \overline{|a|}.$$

Each $\hat{U} \in \hat{\mathcal{U}}_o$ is solid.

Proof. It is clear that $(\hat{\mathbf{B}}_o, \hat{+}, \hat{\cdot})$ is a vector space over $\mathbb{R}$, so we start by proving that $\hat{\mathcal{U}}_o$ is a neighborhood base of $\bar{0}$. For each $U \in \mathcal{U}_o$, $0 + \tilde{U} \subseteq \tilde{U}$, so $\bar{0} \in \hat{U}$. Given $U \in \mathcal{U}_o$ and $a \in \mathbf{B}_o$ with $\bar{a} \in \hat{U}$, there is a $W \in \mathcal{U}_o$ with $|a| + \tilde{W} \subseteq \tilde{U}$. Fix $\bar{b} \in \bar{a} + \hat{W}$. Then $b = a + c + i$, where $\overline{|c|} \subseteq \tilde{W}$ and $i \approx 0$. It follows that for any $j \approx 0$,

$$||b| + j| \leq |a| + |c| + |i| + |j| \in |a| + \tilde{W} \subseteq \tilde{U},$$

and so $\bar{b} \in \hat{U}$ since $\tilde{U}$ is solid. Therefore, $\bar{a} + \hat{W} \subseteq \hat{U}$; that is, $\hat{U}$ is a neighborhood of each of its points.

To show that $\hat{\mathcal{U}}_o$ defines a Hausdorff topology, we fix an $\bar{a} \in \hat{\mathbf{B}}_o$ such that $(\bar{a} \hat{+} \hat{U}) \cap \hat{U} \neq \emptyset$ for each $\hat{U} \in \hat{\mathcal{U}}_o$; we must show that $\bar{a} = \bar{0}$. For each $U \in \mathcal{U}_o$, there are points b and b' such that $b \approx b'$, $b = a + c$ where $|c| \in U$, and $|b'| \in U$. By saturation, we may assume that $b \approx a$ and $b' \approx 0$, whence $a \approx 0$; i.e., $\bar{a} = \bar{0}$. To show that $\hat{+}$ and $\hat{\cdot}$ are continuous operations, we fix $\lambda \in \mathbb{R}$ and a, $b \in \mathbf{B}$ with $\bar{a} \cap \mathbf{B}_o \neq \emptyset$ and $\bar{b} \cap \mathbf{B}_o \neq \emptyset$. Fix $\hat{U} \in \hat{\mathcal{U}}_o$ and $\hat{W} \in \hat{\mathcal{U}}_o$ with $\hat{W} + \hat{W} \subseteq \hat{U}$. Then

$$(\bar{a} + \hat{W}) + (\bar{b} + \hat{W}) \subseteq (\bar{a} + \bar{b}) + \hat{U}.$$

Also, there is a $V \in \mathcal{U}_o$ with $V + V + (|\lambda| + 1) \cdot V \subseteq U$. By saturation and Part 2 of Lemma 2.2, there is an $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1$ such that if $|\lambda - \lambda'| < \varepsilon$, then $|\lambda - \lambda'|\,|a| \in V$. Fix $\lambda' \in \mathbb{R}$ with $|\lambda - \lambda'| < \varepsilon$ and fix $\bar{a}' \in \bar{a} + \hat{V}$. Then $|a - a'| \in V$, and for any $i \approx 0$,

$$||\lambda \cdot a - \lambda' \cdot a'| + i| \leq |\lambda - \lambda'|\,|a| + |\lambda'|\,|a - a'| + |i| \in U,$$

whence $\lambda' \hat{\cdot} \bar{a}' \in \lambda \hat{\cdot} \bar{a} + \hat{U}$. Thus, our space satisfies Property A for a topological vector lattice. Property B follows from Lemmas 2.2 and 2.3.

To establish Property C, we fix a, $b \in \mathbf{B}$ with $\bar{a} \cap \mathbf{B}_o \neq \emptyset$ and $\bar{b} \cap \mathbf{B}_o \neq \emptyset$. By Lemma 2.2 and the properties of $\mathbf{B}_o$, $\overline{a \vee b} \cap \mathbf{B}_o \neq \emptyset$. Since $a \leq a \vee b$ and $b \leq a \vee b$, $\bar{a} \mathrel{\hat{\leq}} \overline{a \vee b}$ and $\bar{b} \mathrel{\hat{\leq}} \overline{a \vee b}$. If $\bar{a} \mathrel{\hat{\leq}} \bar{c}$ and $\bar{b} \mathrel{\hat{\leq}} \bar{c}$ for some $\bar{c} \in \hat{\mathbf{B}}_o$, then by Lemma 2.3, there are elements c, $c' \in \bar{c}$ with $a \leq c$ and $b \leq c'$, whence $a \vee b \leq c \vee c' \approx c$ and $\overline{a \vee b} \mathrel{\hat{\leq}} \bar{c}$. Thus, $\overline{a \vee b} = \bar{a} \vee \bar{b}$. Similarly $\overline{a \wedge b} = \bar{a} \wedge \bar{b}$, and since $-\bar{a} = \overline{-a}$, $\overline{|a|} = |\bar{a}|$.

Fix $\hat{U} \in \hat{\mathcal{U}}_o$ and points a, $b \in \mathbf{B}$ with $\bar{a} \cap \mathbf{B}_o \neq \emptyset$ and $\bar{b} \cap \mathbf{B}_o \neq \emptyset$. To show that $\hat{U}$ is solid, we suppose that $|\bar{b}| \mathrel{\hat{\leq}} |\bar{a}|$ and $\bar{a} \in \hat{U}$. Then $|\bar{a}| \in \hat{U}$ since $|\bar{a}| = \overline{|a|}$ and $\overline{|a|} \subseteq U$. By Lemma 2.3, for any $i \approx 0$, there is a $c \approx |a|$ (whence $c \in U$) with $||b| + i| \leq c$. Since U is semi-solid, $|b| + i \in U$. It follows that $\bar{b} \in \hat{U}$. □

We will call $(\hat{\mathbf{B}}_o, \hat{+}, \hat{\cdot}, \hat{\mathcal{U}}_o, \hat{\leq})$ the $\mathcal{U}_o$**-nonstandard hull of** $\mathbf{B}_o$. We will write $+, \cdot$, and $\leq$ for the operations and order relation in $\hat{\mathbf{B}}_o$ when the meaning is clear. When $\mathbf{B}_o = \mathcal{U}_o$-Lmd$\mathbf{B}$, we may also write $\hat{\mathbf{B}}$

instead of $\hat{\mathbf{B}}_o$. We end this section by establishing the equivalence of suprema and limits of increasing sequences in $\hat{\mathbf{B}}_o$.

Proposition 2.6. *Let $\langle \bar{a}_n \rangle_{n\in\mathbb{N}}$ be an increasing sequence in $\hat{\mathbf{B}}_o$. Given an element $\bar{a} \in \hat{\mathbf{B}}_o$, the following statements are equivalent.*
(1) $\lim_{n\to\infty} \bar{a}_n = \bar{a}$.
(2) $\sup_{n\in\mathbb{N}} \bar{a}_n$ *exists in* $\hat{\mathbf{B}}_o$ *and equals* $\bar{a}$.

Proof. Assume (1) holds. If $\bar{c} \in \hat{\mathbf{B}}_o$ is not an upper bound of the sequence $\langle \bar{a}_n \rangle$, then there is an $n \in \mathbb{N}$ and a $U \in \mathcal{U}_o$ such that $(\bar{a}_n \vee \bar{c}) - \bar{c} \notin \hat{U}$. Since $\hat{U}$ is solid, $(\bar{a}_m \vee \bar{c}) - \bar{c} \notin \hat{U}$ for all $m \geq n$ in $\mathbb{N}$, whence $\lim_{n\to\infty}((\bar{a}_n \vee \bar{c}) - \bar{c}) \neq \bar{0}$. Now, for each $n \in \mathbb{N}$,

$$0 \leq (\bar{a}_n \vee \bar{a}) - \bar{a} \leq |\bar{a}_n - \bar{a}| \,,$$

so $\bar{a}$ must be an upper bound of $\langle \bar{a}_n \rangle$. If $\bar{b}$ is also an upper bound of the sequence, then for all $n \in \mathbb{N}$,

$$0 \leq \left(\bar{a} \vee \bar{b}\right) - \bar{b} \leq \bar{a} - \bar{a}_n = |\bar{a} - \bar{a}_n| \,.$$

Taking the limit, we have $\left(\bar{a} \vee \bar{b}\right) - \bar{b} = 0$; i.e., $\bar{a} \mathrel{\hat{\leq}} \bar{b}$.

Now assume (2) holds. Because $\mathbf{B}_o$ is closed with respect to the operations $\vee$ and $\wedge$, we may assume that a and all a_n are in $\mathbf{B}_o$ and that $a_1 \leq \cdots \leq a_n \leq \cdots \leq a$. If we do not have $\lim_{n\to\infty} \bar{a}_n = \bar{a}$, then there is a $U \in \mathcal{U}_o$ such that $a - a_n \notin U$ for all $n \in \mathbb{N}$. By saturation, there then exists an element $b \in \mathbf{B}$ with $a_n \leq b \leq a$ for all $n \in \mathbb{N}$ and $a - b \notin U$. Since $\mathbf{B}_o$ is nearly interval closed, there is a $b' \approx b$ in $\mathbf{B}_o$. But this is impossible, since then $\bar{b}$ is an upper bound of the sequence $\langle \bar{a}_n \rangle$ in $\hat{\mathbf{B}}_o$, and $\bar{b}$ is strictly smaller than $\bar{a}$. □

3. An Integral Satisfying the Monotone Convergence Theorem

Let $(\mathbf{D}, +, \cdot, \mathcal{D}, \leq)$ be a second internal topological vector lattice with nonstandard hull $(\hat{\mathbf{D}}_o, \hat{+}, \hat{\cdot}, \hat{\mathcal{D}}_o, \hat{\leq})$ with respect to an $\mathbb{R}$-neighborhood base $\mathcal{D}_o \subseteq \mathcal{D}$ and an $\mathbb{R}$-sublattice $\mathbf{D}_o$ of $\mathbf{D}$. When we write $\hat{\mathbf{D}}$ instead of $\hat{\mathbf{D}}_o$ we mean the nonstandard hull formed from $\mathbf{D}_o = \mathcal{D}_o$-Lmd$\mathbf{D}$. We will write $+, \cdot$, and $\leq$ for the operations and order relation in $\hat{\mathbf{D}}_o$ when there is no risk of confusion. Fix a nonempty internal set X. The ordering $\leq$ and the lattice operations can be applied to both internal and external functions mapping X into $\mathbf{B}$. For example, $f \vee g$ is defined at each $x \in X$ by setting $f \vee g(x) = f(x) \vee g(x)$.

We now fix an internal vector lattice L of functions mapping X into $\mathbf{B}$. That is, L is an internal vector space over ${}^*\mathbb{R}$ consisting of $\mathbf{B}$-valued functions such that for each φ, $\psi \in L$, the functions $\varphi \vee \psi$ and $\varphi \wedge \psi$ are again in L. We also fix an internal positive linear operator I mapping L into $\mathbf{D}$; we call $I(\varphi)$ the **internal integral** of φ.

An internal or external function $h : X \to \mathbf{B}$ is called a **null function** if for any $V \in \mathcal{D}_o$ there is a $\varphi \in L$ with $|h| \leq \varphi$ and $I(\varphi) \in V$. The set of null functions on X is denoted by L_0. We let L_1 denote the set of all $\hat{\mathbf{B}}_o$-valued functions f on X for which there exists a pair $\varphi \in L$, $h \in L_0$ with the property that for each $x \in X$,

$$\overline{\varphi + h}(x) := \overline{\varphi(x) + h(x)} = f(x).$$

Here, we say that $\varphi \in L$ is an **internal representative** of $f \in L_1$.

As an example, we note first that the constant function 0 is in L_0. It follows therefore that if g is in L and $g : X \to \mathbf{B}_o$, then $\bar{g}$ is in L_1. We now show that L_0 and L_1 are vector lattices.

Proposition 3.1. *The sets L_0 and L_1 are vector lattices over $\mathbb{R}$. Indeed, if φ and ψ are internal representatives respectively of f and g in L_1, then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are internal representatives respectively of $f \vee g$ and $f \wedge g$. Moreover, for any countable set $\{h_n : n \in \mathbb{N}\}$ contained in L_0, there is an $h \in L_0$ such that $|h_n| \leq h$ for all $n \in \mathbb{N}$.*

Proof. It is easy to see that both L_0 and L_1 are vector spaces over $\mathbb{R}$ and L_0 is a lattice. To show that L_1 is a lattice, fix f, $g \in L_1$, and also fix φ, $\psi \in L$, and h, $j \in L_0$ so that at each $x \in X$, $f(x) = \overline{\varphi + h}(x)$ and $g(x) = \overline{\psi + j}(x)$. Set

$$k := ((\varphi + h) \vee (\psi + j)) - (\varphi \vee \psi).$$

Pointwise, $f \vee g = \overline{(\varphi \vee \psi) + k}$. Therefore, to show that $f \vee g$ is in L_1 and has internal representative $\varphi \vee \psi$, we need only show that $k \in L_0$. To do this, we fix $p := |h| + |j|$ in L_0. Since

$$-p = ((\varphi - p) \vee (\psi - p)) - (\varphi \vee \psi) \leq k \leq ((\varphi + p) \vee (\psi + p)) - (\varphi \vee \psi) = p,$$

k is in L_0. A similar proof shows that $f \wedge g$ is in L_1 and has internal representative $\varphi \wedge \psi$.

We next fix a countable subset $\{h_n : n \in \mathbb{N}\}$ of L_0; we may assume that for each n, $0 \leq h_n \leq h_{n+1}$. Fix $V \in \mathcal{D}_o$. By definition, for each $n \in \mathbb{N}$ there is a $\varphi_n \in L$ with $h_n \leq \varphi_n$ and $I(\varphi_n) \in V$. We may even assume that for each n, $\varphi_n \leq \varphi_{n+1}$. (To see this, choose $W_1 \in \mathcal{D}_o$ and $\psi_1 \in L$ with $W_1 + W_1 \subseteq V$, $h_1 \leq \psi_1$, and $I(\psi_1) \in W_1$; choose $W_2 \in \mathcal{D}_o$ and $\psi_2 \in L$ with $W_2 + W_2 \subseteq W_1$, $h_2 \leq \psi_2$, and $I(\psi_2) \in W_2$, etc. Set $\varphi_n = \Sigma_1^n \psi_i$.) By saturation, there is a $\varphi_V \in L$ with $\varphi_n \leq \varphi_V$ for all $n \in \mathbb{N}$ and $I(\varphi_V) \in V$. Again by saturation, we may at each point

$x \in X$ find a value $h(x)$ such that for each $n \in \mathbb{N}$ and each $V \in \mathcal{D}_o$, $h_n(x) \leq h(x) \leq \varphi_V(x)$. It follows that h is a null function. □

In what follows, we will need a weaker notion of a null function. An internal or external function $h : X \to \mathbf{B}$ is called a **weak null function** if for each $V \in \mathcal{D}_o$ there is a $\varphi \geq 0$ in L with $|h| \tilde{\leq} \varphi$ and $I(\varphi) \in V$. For example, if L is ${}^*c_0$ and $I(\varphi) := \varphi(1)$, then any constant, infinitesimal but non-zero function is a weak null function not in L_0. We denote the class of weak null functions by $\tilde{L}_0$. Proposition 3.4 below shows that if we assume the internal integral I is what we shall call "$\mathcal{D}_o$-continuous", then we do not change the class L_1 by replacing L_0 with $\tilde{L}_0$ in its definition.

The internal integral I is called **$\mathcal{U}_o, \mathcal{D}_o$-continuous** or just, for simplicity, **$\mathcal{D}_o$-continuous**, if for each $\varphi \in L$ with $\varphi \approx 0$ (pointwise) we have $I(\varphi) \approx 0$. This notion is motivated in part by the following consideration. Given $f = \overline{\varphi + h}$ in L_1, we want to use $I(\varphi)$ to define an integral of f. To do so, we need to know that $\overline{I(\varphi)} \in \hat{\mathbf{D}}$ and that $I(\varphi) \approx I(\psi)$ when f also equals $\overline{\psi + j}$, that is, when $\varphi - \psi = h - j + \varepsilon$ where $\varepsilon \approx 0$ pointwise on X. For this, we need to know that $I(\lambda) \approx 0$ when $\lambda \in L \cap \tilde{L}_0$. We now show that this latter property holds if and only if I is $\mathcal{D}_o$-continuous, and that this in turn is a property that corresponds to the finiteness of the measure space in the setting of [12] and [15].

Proposition 3.2. *The following statements are equivalent:*

(a) I is $\mathcal{D}_o$-continuous (i.e., $\varphi \approx 0 \Rightarrow I(\varphi) \approx 0$.)

(b) $I(\varphi)$ is $\mathcal{D}_o$-limited for every pointwise $\mathcal{U}_o$-limited function $\varphi \in L$.

(c) If $\varphi \tilde{\leq} \psi$, then $I(\varphi) \tilde{\leq} I(\psi)$.

(d) $I(\varphi) \approx 0$ for every $\varphi \in L \cap \tilde{L}_0$.

Proof. To show that (a) implies (b), assume that I is $\mathcal{D}_o$-continuous and $\varphi \in L$ is pointwise $\mathcal{U}_o$-limited. Given any $H \in {}^*\mathbb{N}_\infty$, it now follows from Lemma 2.1 that $(1/H)I(\varphi) = I((1/H)\varphi) \approx 0$, and thus $I(\varphi)$ is a $\mathcal{D}_o$-limited. To show that (b) implies (a), fix a pointwise infinitesimal function $\varphi \in L$. For each $U \in \mathcal{U}_o$ and $n \in \mathbb{N}$, $n \cdot |\varphi(x)| \in U$ for all $x \in X$. By saturation, there is for each $U \in \mathcal{U}_o$ an $H_U \in {}^*\mathbb{N}_\infty$ with $H_U \cdot |\varphi(x)| \in U$ for all $x \in X$. Again by saturation, there is an $H \in {}^*\mathbb{N}_\infty$ such that $H \leq H_U$ for all $U \in \mathcal{U}_o$, whence $H \cdot \varphi$ is pointwise $\mathcal{U}_o$-limited. By assumption, $H \cdot I(\varphi) = I(H \cdot \varphi)$ is $\mathcal{D}_o$-limited, and so by Lemma 2.1, $I(\varphi) \approx 0$.

To show that (a) implies (c), assume that I is $\mathcal{D}_o$-continuous and $\varphi \tilde{\leq} \psi$ in L. Since $(\varphi \vee \psi) - \psi \approx 0$, $I(\psi) \approx I(\varphi \vee \psi)$. By Lemma 2.2

and the positivity of the internal integral I,

$$I(\varphi) \vee I(\psi) \approx I(\varphi) \vee I(\varphi \vee \psi) = I(\varphi \vee \psi) \approx I(\psi).$$

That is, $I(\varphi) \tilde{\lesssim} I(\psi)$. To show (c) implies (d), fix $\varphi \in L \cap \tilde{L}_0$. For each $V \in \mathcal{D}_o$ there is an $\psi_V \geq 0$ in L with $|\varphi| \tilde{\lesssim} \psi_V$ and $I(\psi_V) \in V$; in particular, for each $U \in \mathcal{U}_o$, $((|\varphi| \vee \psi_V) - \psi_V)(x) \in U$ for all $x \in X$. By saturation, there is a $\psi \geq 0$ in L with $|\varphi| \tilde{\lesssim} \psi$ and $I(\psi) \approx 0$. Since

$$|I(\varphi)| \leq I(|\varphi|) \tilde{\lesssim} I(\psi) \approx 0,$$

we are done. It is clear that (d) implies (a), since if $\varphi \approx 0$, then $|\varphi| \tilde{\lesssim} 0$. □

From this point on, we assume that I is $\mathcal{D}_o$-continuous. The following two results show that internal representatives of L_1 functions have $\mathcal{D}_o$-limited integrals, and we cannot augment the class L_1 by replacing L_0 with $\tilde{L}_0$.

Proposition 3.3. *If $\varphi \in L$, $h \in L_0$, and $\varphi + h$ is pointwise $\mathcal{U}_o$-limited, then $I(\varphi)$ is $\mathcal{D}_o$-limited.*

Proof. Fix $V \in \mathcal{D}_o$; we must show that $|I(\varphi)| \in n \cdot V$ for some $n \in \mathbb{N}$. Fix $W \in \mathcal{D}_o$ with $W + W + W + W \subset V$. Choose $\psi \in L$ with $|h| \leq \psi$ and $I(\psi) \in W$. Note that $I(\psi) = |I(\psi)|$. Since

$$\varphi - \psi \leq \varphi + h \leq \varphi + \psi,$$

we have

$$0 \leq (\varphi - \psi) \vee 0 \leq (\varphi + h) \vee 0$$

and

$$(\varphi + h) \wedge 0 \leq (\varphi + \psi) \wedge 0 \leq 0.$$

Thus $(\varphi - \psi) \vee 0$ and $(\varphi + \psi) \wedge 0$ are pointwise $\mathcal{U}_o$-limited. By Proposition 3.2, there is an $n \in \mathbb{N}$ such that

$$(1/n) I((\varphi - \psi) \vee 0) \in W \quad \text{and} \quad (-1/n) I((\varphi + \psi) \wedge 0) \in W.$$

Now

$$(\varphi + \psi) \wedge 0 \leq \varphi + \psi = (\varphi - \psi) + 2\psi \leq ((\varphi - \psi) \vee 0) + 2\psi,$$

so

$$[(\varphi + \psi) \wedge 0] - \psi \leq \varphi \leq [(\varphi - \psi) \vee 0] + \psi.$$

It follows that

$$|I(\varphi)| \leq -I((\varphi + \psi) \wedge 0) + I(\psi) + I((\varphi - \psi) \vee 0) + I(\psi) \in n \cdot V.$$

Since $n \cdot V$ is semi-solid, $|I(\varphi)| \in n \cdot V$. □

Proposition 3.4. *For each $g \in \tilde{L}_0$, there is an $h \in L_0$ with $h \approx g$ pointwise. Thus, if f is a $\hat{B}_o$-valued function on X with $f = \overline{\varphi + g}$ for $\varphi \in L$ and $g \in \tilde{L}_0$, then $f \in L_1$.*

Proof. Fix $g \in \tilde{L}_0$. For each $V \in \mathcal{D}_o$, choose $\varphi_V \geq 0$ in L with $|g| \tilde{\leq} \varphi_V$ and $I(\varphi_V) \in V$. Given a finite set $\{V_1, \cdots, V_n\} \subset \mathcal{D}_o$ and setting

$$g' := (-\varphi_{V_1} \vee \cdots \vee -\varphi_{V_n} \vee g) \wedge \varphi_{V_1} \wedge \cdots \wedge \varphi_{V_n},$$

we have $g' \approx g$ and $\varphi_{V_i} \leq g' \leq \varphi_{V_i}$ for $1 \leq i \leq n$. Thus, given $x \in X$ and a finite set $\{U_1, U_2, \cdots, U_m\} \subset \mathcal{U}_o$, there is an element $a \in \mathbf{B}$ such that $|a| \leq \varphi_{V_i}(x)$, $1 \leq i \leq n$, and $|g(x) - a| \in U_j$, $1 \leq j \leq m$. By saturation, we may choose a value $h(x)$ with $|h(x)| \leq \varphi_V(x)$ and $|g(x) - h(x)| \in U$ for all $V \in \mathcal{D}_o$, $U \in \mathcal{U}_o$. The function h is in L_0 and $h \approx g$. □

As in [15], we can characterize elements of L_1 in terms of approximations from above and below by elements of L.

Proposition 3.5. *Given $g : X \to \mathbf{B}_o$, $\bar{g}$ is in L_1 iff for each $V \in \mathcal{D}_o$ there are functions φ and ψ in L with $\varphi \leq \psi$, $\varphi \tilde{\leq} g \tilde{\leq} \psi$ and $I(\psi - \varphi) \in V$.*

Proof. Assume that $\bar{g} \in L_1$. Then $g \approx \varphi + h$ for some $\varphi \in L$ and $h \in L_0$. Fix $V \in \mathcal{D}_o$. There is a $W \in \mathcal{D}_o$ with $W + W \subseteq V$ and a function $\psi \in L$ such that $|h| \leq \psi$ and $I(\psi) \in W$. Therefore, $-\psi + \varphi \tilde{\leq} g \tilde{\leq} \psi + \varphi$ and $I((\psi + \varphi) - (-\psi + \varphi)) \in V$. To establish the converse, we choose for each $V \in \mathcal{D}_o$ functions φ_V and ψ_V in L with $\varphi_V \leq \psi_V$, $\varphi_V \tilde{\leq} g \tilde{\leq} \psi_V$ and $I(\psi_V - \varphi_V) \in V$. Given $U \in \mathcal{U}_o$ and a finite set $\{V_1, V_2, \cdots, V_n\} \subset \mathcal{D}_o$, there is a φ in L (equal to $\varphi_{V_1} \vee \varphi_{V_2} \vee \cdots \vee \varphi_{V_n}$) such that for each i, $1 \leq i \leq n$, each $V \in \mathcal{D}_0$, and all $x \in X$, we have $((\varphi_{V_i} \vee \varphi) - \varphi)(x) \in U$, and $((\psi_V \vee \varphi) - \psi_V)(x) \in U$. By saturation, there is a $\varphi \in L$ such that for every $V \in \mathcal{D}_o$, $\varphi_V \tilde{\leq} \varphi \tilde{\leq} \psi_V$, whence

$$\varphi_V - \psi_V \tilde{\leq} g - \varphi \tilde{\leq} \psi_V - \varphi_V.$$

It follows that $g - \varphi \in \tilde{L}_0$. By Proposition 3.4, $\bar{g} = \overline{\varphi + (g - \varphi)} \in L_1$. □

We can now define a mapping J on L_1 as follows: For each $f \in L_1$, choose an internal representative $\varphi \in L$. (I.e., $f = \overline{\varphi + h}$ for some $h \in L_0$.) Set

$$J(f) := \overline{I(\varphi)};$$

$J(f)$ is called the **integral** of f. By Propositions 3.2, and 3.3, J is a well-defined linear mapping of L_1 into $\hat{\mathbf{D}}$. By Proposition 3.1, if $\varphi \in L$ and $h \in L_0$ with $\overline{\varphi + h} \in L_1$, and $\bar{0} \hat{\leq} \overline{\varphi + h}$, then there exists a $k \in L_0$ with $(\varphi \vee 0) + k \approx \varphi + h$. It follows that the mapping J is positive.

At times, one can improve closure properties by replacing $\mathcal{D}_o$-Lmd $\mathbf{D}$ with an $\mathbb{R}$-sublattice $\mathbf{D}_o \subset \mathcal{D}_o$-Lmd $\mathbf{D}$. For example, one may need such a restriction to obtain a σ-algebra for the domain of a measure related to the integral J (see Proposition 4.2). We also use such a restriction in our discussion of the Bochner Integral in Section 7.

Fix an $\mathbb{R}$-sublattice $\mathbf{D}_o \subseteq \mathcal{D}_o$-Lmd $\mathbf{D}$, and let L_1° be the subspace of L_1 defined by setting

$$L_1^\circ := \{f \in L_1 : J(f) \in \hat{\mathbf{D}}_o\}.$$

The following, principal result of this section is given in terms of the space L_1°; the reader may, of course, specialize to the case that $\mathbf{D}_o = \mathcal{D}_o$-Lmd $\mathbf{D}$ and $L_1^\circ = L_1$.

Theorem 3.6 (Monotone Convergence Theorem)**.** *Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence in L_1°.*

(a) If there is a least upper bound f of the family $\langle f_n \rangle$ in L_1°, then $\sup J(f_n)$ *exists in $\hat{\mathbf{D}}_o$, and $J(f) = \sup J(f_n)$.*

(b) If for each $x \in X$, $f(x) := \sup\ f_n(x)$ exists in $\hat{\mathbf{B}}_o$, then $f \in L_1^\circ$ if and only if $\sup J(f_n)$ *exists in $\hat{\mathbf{D}}_o$. In this case, f is the least upper bound of the family $\langle f_n \rangle$ in L_1°, and so $J(f) = \sup\ J(f_n)$.*

Proof. We may assume that each $f_n \geq 0$ by replacing f_n with $\overline{f_n - f_1}$. For each $n \in \mathbb{N}$, we fix $\varphi_n \in L$ and $h_n \in L_0$ so that $f_n = \overline{\varphi_n + h_n}$. By Proposition 3.1, we may assume that $0 \leq \varphi_n \leq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and that there is an $h \in L_0$ such that for each $n \in \mathbb{N}$, $|h_n| \leq h$. To establish Part a, we assume that the family $\langle f_n \rangle$ has a least upper bound in L_1° of the form $f = \overline{\varphi + j}$ with $\varphi \in L$ and $j \in L_0$. By Proposition 3.1, we may again modify the φ_n and then, after extending the sequence, choose an $\omega \in {}^*\mathbb{N}_\infty$ so that for all $n \in \mathbb{N}$ $\quad \varphi_n \leq \varphi_{n+1} \leq \varphi_\omega \leq \varphi$, and thus $J(f_n) \hat{\leq} \overline{I(\varphi_\omega)} \hat{\leq} \overline{I(\varphi)}$. Given any upper bound $\bar{b}$ of the family $\{J(f_n) : n \in \mathbb{N}\}$, we may choose φ_ω so that $\overline{I(\varphi_\omega)} \hat{\leq} \bar{b}$. Therefore, we can prove $J(f) = \overline{I(\varphi)}$ is the least such upper bound by showing that $I(\varphi - \varphi_\omega) \approx 0$. Now, since $j \wedge 0 \leq (j + (\varphi - \varphi_\omega)) \wedge h \leq h$,the function $k := h \wedge (\varphi - \varphi_\omega + j) \in L_0$. Moreover,

$$\varphi_\omega + k \leq \varphi_\omega + (\varphi - \varphi_\omega + j) = \varphi + j.$$

Also, for each $n \in \mathbb{N}$,

$$\varphi_n + h_n \mathrel{\tilde{\leq}} \varphi_\omega + (\varphi - \varphi_\omega + j),$$

and

$$\varphi_n + h_n \leq \varphi_\omega + h,$$

so

$$\varphi_n + h_n \mathrel{\tilde{\leq}} \varphi_\omega + k \leq \varphi + j.$$

Since $\mathbf{B}_o$ and $\mathbf{D}_o$ are nearly interval closed, $\overline{\varphi_\omega + k} \in L_1^\circ$, and therefore

$$\varphi + j \mathrel{\tilde{\leq}} \varphi_\omega + k$$

since f is the least upper bound of the f_n's in L_1°. We now have

$$0 \leq \varphi - \varphi_\omega \mathrel{\tilde{\leq}} k - j,$$

whence $\varphi - \varphi_\omega \in \tilde{L}_0$, and thus by Proposition 3.2, $I(\varphi - \varphi_\omega) \approx 0$.

For Part b, we assume that for each $x \in X$, $f(x) := \sup f_n(x)$ exists in $\hat{\mathbf{B}}_o$. If $f \in L_1^\circ$, then f is the least upper bound of the family $\langle f_n \rangle_{n\in\mathbb{N}}$ in L_1°, and so by Part a, $\sup J(f_n)$ exists in $\hat{\mathbf{D}}_o$. For the converse, we assume that $\bar{a} := \sup J(f_n)$ exists in $\hat{\mathbf{D}}_o$; we must show that $f \in L_1^\circ$. Since $((I(\varphi_n) \vee a) - a) \in V$ for each $V \in \mathcal{D}_o$ and $n \in \mathbb{N}$, there exists by saturation a $\varphi_\omega \in L$ such that for all $n \in \mathbb{N}$, $\varphi_n \leq \varphi_\omega$ and $I(\varphi_n) \leq I(\varphi_\omega) \mathrel{\tilde{\leq}} a$. Since $\mathbf{D}_o$ is nearly interval closed, $\overline{I(\varphi_\omega)} \in \hat{\mathbf{D}}_o$. Therefore, by the definition of $\bar{a}$, $\overline{I(\varphi_\omega)} = \bar{a}$. For each $x \in X$, choose $p(x)$ and $j(x)$ in $\mathbf{B}$ so that $\overline{p(x)} = f(x)$ and $\varphi_\omega + j = p$. To show now that $f \in L_1^\circ$, we need only show that $j \in \tilde{L}_0$. For that proof, we fix U and V in $\mathcal{D}_o$ with $U + U \subseteq V$, and we choose $\psi \in L$ with $0 \leq h \leq \psi$ and $I(\psi) \in U$. For each $n \in \mathbb{N}$,

$$\varphi_n + h_n \mathrel{\tilde{\leq}} p \mathrel{\tilde{\leq}} \varphi_\omega + h,$$

so

$$j = p - \varphi_\omega \mathrel{\tilde{\leq}} \varphi_\omega + h - \varphi_\omega = h,$$

and

$$-j = \varphi_\omega - p \mathrel{\tilde{\leq}} \varphi_\omega - \varphi_n - h_n,$$

whence $|j| \tilde{\leq} \varphi_\omega - \varphi_n + \psi$. By Proposition 2.6, we may choose $n \in \mathbb{N}$ so that $I(\varphi_\omega) - I(\varphi_n) \in U$, and thus $I(\varphi_\omega - \varphi_n + \psi) \in U + U \subseteq V$. Since V is arbitrary in $\mathcal{D}_o$, $j \in \tilde{L}_0$. □

4. **Scalar Functions and Vector Measures**

In the real valued Daniell-Stone integration theory, the monotone convergence theorem is used to construct measures. In this section we outline a similar construction. Although more general results are valid, we consider here only the case where $\mathbf{B}$ equals the nonstandard extension ${}^*\mathbb{R}$ of the real numbers $\mathbb{R}$. We let $\mathcal{U}$ be the extension of the usual neighborhood base at 0, i.e.,

$$\mathcal{U} = \{(-a, a) \subseteq {}^*\mathbb{R} : a \in {}^*\mathbb{R},\ a > 0\},$$

and we let $\mathcal{U}_o$ be the collection of standard elements of $\mathcal{U}$, i.e.,

$$\mathcal{U}_o = \{(-a, a) \subseteq {}^*\mathbb{R} : a \in \mathbb{R},\ a > 0\}.$$

We work with the nonstandard hull $\hat{\mathbf{B}}$, and following convention, we identify $\hat{\mathbf{B}}$ with $\mathbb{R}$.

For each $A \subseteq X$, we write 1_A for the characteristic function of A. We will assume that I and the sets L, $\mathbf{D}$, $\mathbf{D}_o$, $\mathcal{D}$ and $\mathcal{D}_o$ defined in Section 3 have been chosen so that 1_X is in L and $I(1_X)$ is in $\mathbf{D}_o$.

Now the functions in L and L_0 take their values in ${}^*\mathbb{R}$, and the functions in L_1 take their values in $\mathbb{R}$. Since 1_X is in L and $I(1_X) \in \mathbf{D}_o$, any function, taking only infinitesimal values, is in L_0. It follows that a real valued function f is in L_1 if and only if there exists a $\varphi \in L$ and an $h \in L_0$ with, $f = \varphi + h$.

Since $\mathbf{D}_o$ is nearly interval closed, and by assumption, $1_X \in L_1^\circ$,

$$\Sigma := \{A \subseteq X : 1_A \in L_1\} = \{A \subseteq X : 1_A \in L_1^\circ\}.$$

In particular, if 1_A is in L, then $A \in \Sigma$. For each $A \in \Sigma$, we set $\mu(A) = J(1_A)$. It is clear that Σ is an algebra. We now show that μ is σ-additive on Σ and that Σ in special cases is even a σ-algebra. Recall that μ is complete on Σ if when A is a subset of $B \in \Sigma$ and $\mu(B) = 0$, then it follows that $A \in \Sigma$. Also recall that a vector lattice Y is called **σ-Dedekind complete** if every increasing sequence of nonnegative elements with an upper bound in Y has a least upper bound in Y.

Proposition 4.1. *The set function μ is complete and σ-additive on the algebra Σ. Moreover, Σ is a σ-algebra if either $\hat{\mathbf{D}}_o$ is σ-Dedekind complete or if L_1° is σ-Dedekind complete.*

Proof. It is clear that μ is additive. If $\langle A_n \rangle_{n\in\mathbf{N}}$ is an increasing sequence in Σ and $A := \cup A_n$ is also in Σ, then 1_A is pointwise in $\mathbb{R}$, and $1_A \in L_1$. By Proposition 2.6 and Theorem 3.6,

$$\lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbb{N}} J(1_{A_n}) = J(1_A) = \mu(A).$$

If $B \in \Sigma$ and $\mu(B) = 0$, then $1_B = \varphi + h$ where $h \in L_0$, $\varphi \in L$, $0 \leq \varphi$, and $I(\varphi) \approx 0$. It follows that $1_B \in L_0$ and for each $A \subseteq B$, $1_A \in L_0$. Therefore, $1_A = 0 + 1_A$ is in L_1, and so $A \in \Sigma$.

We assume now that $\hat{\mathbf{D}}_o$ is σ-Dedekind complete and show that Σ is a σ-algebra; the proof for the case that L_1° is σ-Dedekind complete is similar. Let $\langle B_n \rangle_{n\in\mathbb{N}}$ be an increasing sequence in Σ with union $B \subseteq X$. Then sup $1_{B_n} = 1_B$, and sup $J(1_{B_n})$ exists in $\hat{\mathbf{D}}_o$. By Theorem 3.6 b, $1_B \in L_1^\circ$, and therefore $B \in \Sigma$. □

We conclude this section by stating two results which extend well known results for scalar measure theory (e.g., [12] and [3]).

Theorem 4.2. *Assume that $\hat{\mathbf{D}}_o$ is σ-Dedekind complete. Let $\mathfrak{R}$ be an internal algebra of subsets of X, and let ν be an internal, $\mathbf{D}$-valued, finitely additive measure on $(X, \mathfrak{R})$ such that $\nu(X) \in \mathbf{D}_o$ and $0 \leq \nu(A)$ for all $A \in \mathfrak{R}$. Let $\bar{\nu}$ denote the $\hat{\mathbf{D}}_o$-valued finitely additive measure on $(X, \mathfrak{R})$ defined by setting $\bar{\nu}(A) = \overline{\nu(A)}$ for all $A \in \mathfrak{R}$. Then $\bar{\nu}$ has a complete σ-additive extension μ with values in $\hat{\mathbf{D}}_o$.*

Proof. To apply our general theory, we let L be the lattice of all internal, $\mathfrak{R}$-measurable functions $\varphi : X \to {}^*\mathbb{R}$ such that the range of φ is a finite or hyperfinite subset of ${}^*\mathbb{R}$. The internal integral I: $L \to \mathbf{D}$ is defined in the obvious way using ν. By assumption, $I(1_X) \in \mathbf{D}_o$. Define L_1°, J, Σ, and μ as above. Then $\mathfrak{R} \subseteq \Sigma$ since for each $A \in \mathfrak{R}$, $1_A \in L_1^\circ$. Moreover, μ is a σ-additive extension of $\bar{\nu}$ since for each $A \in \mathfrak{R}$,

$$\bar{\nu}(A) = \overline{I(1_A)} = J(1_A) = \mu(A).$$

The completeness of μ follows from Proposition 4.1. □

Theorem 4.3. *Assume Y is a Hausdorff space and $(\mathbf{E}, +, \mathcal{W}, \leq)$ is a standard topological vector space over $\mathbb{R}$. Let β be a finitely additive function on an algebra $\mathcal{A}$ containing the open sets of Y with β taking its values in $\mathbf{E}$ and $\beta(A) \geq 0$ for each $A \in \mathcal{A}$. Assume that the following tightness condition holds: For every $A \in \mathcal{A}$ and every $V \in \mathcal{W}$, there is a compact F and open O with $F \subseteq A \subseteq O \subseteq Y$ and $\beta(O \backslash F) \in V$. Then β can be extended to a σ-additive measure with values in the usual nonstandard hull $\hat{\mathbf{E}}$ of $\mathbf{E}$ if $\hat{\mathbf{E}}$ σ-Dedekind complete.*

Proof. Let $\mathbf{D} = {}^*\mathbf{E}$, and let $\mathbf{D}_o$ be the limited elements of ${}^*\mathbf{E}$, so $\hat{\mathbf{D}}_o = \hat{\mathbf{E}}$ is the usual nonstandard hull of $\mathbf{E}$. The assumptions of Theorem 4.2 are satisfied by setting

$$(X, \Re, \nu) = ({}^*Y, {}^*\mathcal{A}, {}^*\beta).$$

Let L, I, L_1°, J, Σ, and μ be defined as in the proof of Theorem 4.2. Since Σ is a σ-algebra, the collection $\mathcal{F} = \{A \subset Y : \mathrm{st}^{-1}[A] \in \Sigma\}$ is also a σ-algebra. For each $A \in \mathcal{F}$, set $\overline{\beta}(A) := \mu(\mathrm{st}^{-1}[A])$; $\overline{\beta}$ is σ-additive and complete. To show that it is an extension of β, we fix an arbitrary $A \in \mathcal{A}$ and show that $\mathrm{st}^{-1}[A] \in \Sigma$ and $\mu(\mathrm{st}^{-1}[A]) = \beta(A)$. For any standard neighborhood V of 0, there is a compact $F \subseteq Y$ and open $O \subseteq Y$ with $F \subseteq A \subseteq O$ and $\beta(O \backslash F) \in V$. It follows that ${}^*F \subseteq \mathrm{st}^{-1}[F] \subseteq \mathrm{st}^{-1}[A] \subseteq {}^*O$, and ${}^*\beta({}^*O \backslash {}^*F) \in {}^*V$. Let φ, g, and ψ denote the characteristic functions respectively of ${}^*F$, $\mathrm{st}^{-1}[A]$, and ${}^*O$. Applying Proposition 3.5, we see that g is in L_1°, whence $\mathrm{st}^{-1}[A] \in \Sigma$. Moreover, identifying $\beta(A)$ with its equivalence class in $\hat{\mathbf{E}}$, we have

$$\mu\left(\mathrm{st}^{-1}[A]\right) = J(g) = \overline{{}^*\beta({}^*A)} = \beta(A). \quad \square$$

Part 2. EXAMPLES

5. Banach Limits

In this section, we use our general theory and a special measure to extend Robinson's construction [23] of Banach limits to vector valued sequences. We will work with the space $\mathbf{E}_b^{\mathbb{N}}$ consisting of all bounded sequences with values in a standard, locally convex, topological vector lattice $\mathbf{E}$. A sequence $\langle x_n : n \in \mathbb{N} \rangle$ in $\mathbf{E}$ is bounded if for each neighborhood U of 0 in $\mathbf{E}$, there is an $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $|x_n| \in m \cdot U$. We will say that a sequence $\langle x_n : n \in \mathbb{N} \rangle$ in $\mathbf{E}$ converges absolutely to $a \in \mathbf{E}$ if $\lim_{n \to \infty} |x_n - a| = 0$. We assume that $\mathbf{E}$ has a base of neighborhoods for 0 formed by a collection $\mathcal{W}$ of convex, semi-solid sets. Therefore, a sequence in $\mathbf{E}$ converges to $a \in \mathbf{E}$ if it converges absolutely to a, and the converse holds if there is a base of solid sets for the neighborhoods of 0. Our construction produces a shift invariant linear mapping on $\mathbf{E}_b^{\mathbb{N}}$ which maps convergent sequences to their limit when they converge absolutely to that limit. With small modifications, our construction also works for sequences in locally convex topological vector spaces that are not lattices.

We set the following correspondence with the notation of our general theory:

$$X = {}^*\mathbb{N},\ \ \mathbf{B} = \mathbf{D} = {}^*\mathbf{E},\ \ \mathcal{U}_o = \mathcal{D}_o = \{{}^*V : V \in \mathcal{W}\},\ \ \mathbf{B}_o = \mathbf{D}_o = \mathrm{Lmd}\,{}^*\mathbf{E}.$$

Fix $H \in \mathbb{N}_\infty$, and let λ be the internal measure on ${}^*\mathbb{N}$ given by $\lambda(i) = 1/H$ for $1 \le i \le H$, $\lambda(i) = 0$ for $i > H$. Let L consist of all internal functions from ${}^*\mathbb{N}$ into ${}^*\mathbf{E}$, and let I be the internal integral on L given by λ. That is, for each $\varphi \in L$,

$$I(\varphi) = (1/H) \cdot \Sigma_{i=1}^{H} \varphi(i).$$

To see that I is $\mathcal{D}_o$-continuous, fix $V \in \mathcal{W}$ and note that if $|\varphi(i)|$ is in ${}^*V$, for all $i \le H$, then the convex combination $I(\varphi)$ has $|I(\varphi)| \in {}^*V$.

For a locally convex $\mathbf{E}$, the integral J of our general theory gives a positive linear mapping from L_1 into $\hat{\mathbf{E}}$. Given $G \in \mathbf{E}_b^{\mathbb{N}}$, the function ${}^*G$ is in L, and so $\overline{{}^*G}$ is in L_1. We define $\hat{J} : \mathbf{E}_b^{\mathbb{N}} \to \hat{\mathbf{E}}$ by setting $\hat{J}(G) = J(\overline{{}^*G})$ for all $G \in \mathbf{E}_b^{\mathbb{N}}$. That is, $\hat{J}$ is the restriction of J to the subspace of L_1 consisting of all functions of the form $\overline{{}^*G}$ where $G \in \mathbf{E}_b^{\mathbb{N}}$. In interpreting the following result, we identify each $a \in \mathbf{E}$ with the corresponding element $\bar{a} \in \widehat{\mathbf{E}}$.

Theorem 5.1. *The mapping $\hat{J}$ is linear and shift invariant on $\mathbf{E}_b^{\mathbb{N}}$. If $G \in \mathbf{E}_b^{\mathbb{N}}$ converges absolutely to $a \in \mathbf{E}$, then $\hat{J}(G) = \bar{a}$.*

Proof. Since J is linear, so is $\hat{J}$. To see that J, and therefore $\hat{J}$, are shift invariant, we fix a pointwise $\mathcal{U}_o$-limited φ in L, and we let S denote the shift operator. Since $S(\varphi)(i) = \varphi(i+1)$ for each $i \in {}^*\mathbb{N}$, it is easy to see that $J(\overline{S(\varphi)}) = J(\bar{\varphi})$. Finally, we fix a $G \in \mathbf{E}_b^{\mathbb{N}}$ converging absolutely to some point $a \in \mathbf{E}$. For each $i \in {}^*\mathbb{N}$, ${}^*G(i) = {}^*a + g(i)$, where $g(i)$ is $\mathcal{U}_o$-limited for every $i \in \mathbb{N}$ and $g(i) \approx 0$ for all $i \in {}^*\mathbb{N}_\infty$. To show that $\hat{J}(G) = \bar{a}$, we need only show that $I(g) \approx 0$. Given $m \le H$, let $g_m(i) = g(i)$ for $1 \le i \le m$, and let $g_m(i) = 0$ for $i > m$. For all $m \in \mathbb{N}$, and therefore for some $m \in {}^*\mathbb{N}_\infty, I(g_m) \approx 0$. For the same unlimited m, $I(g - g_m)$ is a convex combination of infinitesimals and is therefore infinitesimal. It follows that

$$I(g) = I(g - g_m) + I(g_m) \approx 0. \ \square$$

There are some interesting examples of the mapping $\hat{J}$ for the space $\mathbf{E} = \ell^p$, $1 \le p \le +\infty$. Let 1_H denote the internal sequence which is identically equal to 1 for $1 \le i \le H$ and is 0 for $i > H$. Let δ^n denote the sequence which is identically equal to 0 except at n where it is equal to 1. The standard sequence of sequences $\langle \delta^n \rangle = \langle \delta^n \rangle_{n \in \mathbb{N}}$ is a sequence with values in ℓ^p for $1 \le p \le +\infty$.

First, as a counter-example, we note that if we took $p = 1/2$, the sequence $\langle (1/H) \cdot \delta^n \rangle$ would be a sequence with infinitesimal values in the non-locally convex space ℓ^p. For this space, the mapping I would not be $\mathcal{D}_o$-continuous. In particular, for $\langle (1/H) \cdot \delta^n \rangle$ we would have

$$\left\| I\left(\left\langle \frac{1}{H} \cdot \delta^n \right\rangle\right) \right\|_p = \left\| \frac{1}{H^2} \cdot 1_H \right\|_p = 1^2 = 1.$$

For $1 \leq p \leq +\infty$ and the sequence $\langle \delta^n \rangle$, we have

$$\hat{J}\left(\langle \delta^n \rangle\right) = \overline{\frac{1}{H} \cdot 1_H}.$$

If $p > 1$, $\hat{J}\left(\langle \delta^n \rangle\right) = 0$ since the internal norm in ℓ^p of $(1/H) \cdot 1_H$ is $1/H \approx 0$ when $p = +\infty$, and it is $(H/H^p)^{1/p} \approx 0$ when $1 < p < +\infty$. Thus, for example, in ℓ^2 the $\hat{J}$ "limit" of the orthonormal sequence $\langle \delta^n \rangle$ is 0. On the other hand, for $p = 1$, $\hat{J}\left(\langle \delta^n \rangle\right)$ is not 0 since the internal norm of $(1/H) \cdot 1_H$ is 1. The value $\hat{J}\left(\langle \delta^n \rangle\right) = \overline{(1/H) \cdot 1_H}$ is not, however, in ℓ^1 itself. Only in the nonstandard hull of ℓ^1 can one find this general Banach limit of the sequence $\langle \delta^n \rangle$.

6. **An Integral Representation for Harmonic Functions**

A topological vector space where the neighborhoods of 0 are semi-solid but not solid is provided by the space $\mathcal{H}_D$ consisting of difference of positive harmonic functions on the unit disk D. Here, the topology is the topology of uniform convergence on compact sets (the ucc topology). We write $|h|$ for the pointwise absolute value and $|h|_{\mathcal{H}}$ for the lattice absolute value of each element h in the space. A typical neighborhood U_K^ε of the harmonic function 0 is given by a compact set $K \subset D$ and a positive $\varepsilon \in \mathbb{R}$; it consists of all functions h in the space such that $\sup_{x \in K} |h(x)| < \varepsilon$. To demonstrate that such neighborhoods of 0 are not solid, we partition the unit circle C into intervals of length $\pi/2^n$. Starting from the point 1 on C, set $f_n = 1$ on the even numbered intervals and set $f_n = -1$ on the odd numbered intervals. Let h_n denote the harmonic extension of f_n obtained by using the Poisson kernel. The sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ converges to 0 in the ucc topology. On the other hand, for each $n \in \mathbb{N}$, the pointwise absolute value $|h_n|$ of h_n is not harmonic. For each $n \in \mathbb{N}$, the lattice absolute value $|h_n|_{\mathcal{H}}$ is the harmonic extension of $|f_n|$; i.e., $|h_n|_{\mathcal{H}} \equiv 1$ on the disk D. Thus we have an example of a sequence converging to 0 for which the lattice absolute value does not converge to 0. It follows that the neighborhoods U_K^ε of 0 are not solid; they are, however, semi-solid. Notice that if we go to a nonstandard model and let n be unlimited in ${}^*\mathbb{N}$, then h_n is in every standard neighborhood of 0 but the lattice absolute value of h_n is 1. We do not consider such an element to be an infinitesimal.

A metric ρ which generates the ucc topology on $\mathcal{H}_D$ is defined at each h and g in the space by setting

$$\rho(h,g) = \sum_{n=1}^{\infty} \frac{1}{2^n}(\sup\{|h(z) - g(z)| : |z| \leq 1 - \frac{1}{n}\} \wedge 1).$$

It follows from Harnack inequality that the coarsest possible, locally solid refinement of the u.c.c. topology obtained using the method in Section 2 is generated by a metric d defined at each h and g in $\mathcal{H}_D$ by setting $d(h,g) = |h-g|_{\mathcal{H}}(0)$. By the Riesz-Herglotz Theorem, $\mathcal{H}_D$ with this topology is equivalent to the space of finite signed Borel measures on the unit circle with the topology given by the total variation norm. For more details, see [17].

Instead of restricting the discussion in this section to just ordinary harmonic functions on the unit disk, we will consider a harmonic space of functions on a connected and locally connected, locally compact but not compact, open set W. The reader may of course assume that W is an open subset of Euclidean space and "harmonic" has the usual meaning; the canonical example is provided by harmonic functions on the unit disk in the plane. Our results will, however, be valid in quite general potential theoretic settings. We have in mind a Brelot [5] harmonic space as described in [13]. With some modifications, our construction will also be valid for a Bauer [4] harmonic space (see [14]). We need the constant function 1 to be at least superharmonic; for convenience, we assume here that it is harmonic. We also assume that W is σ-compact.

In the nonstandard extension ${}^*W$ of W, we fix an internal, connected open set Ω with $\bar{\Omega} \subset {}^*W$. We assume that the closure $\bar{\Omega}$ of Ω is compact in the internal sense, that the extension of every standard compact subset of W is contained in Ω, and that the boundary of Ω is regular for the internal Dirichlet problem. For the case that W is the unit disk in the plane, we may take $\Omega = \{z \in {}^*\mathbb{C} : |z| < 1 - 1/\eta\}$ for some $\eta \in {}^*\mathbb{N}_\infty$.

For the general case, let X be the boundary of Ω, and let L be the space of internal, continuous ${}^*\mathbb{R}$-valued functions on X. For each $f \in L$, let $I(f)$ be the harmonic extension of f in the region Ω. We set $\mathbf{D}$ equal to the space of differences of internal positive harmonic functions on Ω. For the neighborhood system $\mathcal{D}$ of 0 in $\mathbf{D}$ we take neighborhoods corresponding the internal topology of uniform convergence on compact subsets of Ω. To obtain $\mathcal{D}_o$, we fix a countable exhaustion K_n of compact subsets of W, and for each $n \in \mathbb{N}$ we let U_n be the set of internal harmonic functions h on Ω with $\sup_{x \in {}^*K_n} |h(x)| < 1/n$; we set

$\mathcal{D}_o := \{U_n : n \in \mathbb{N}\}$. Again, we are using the notation $|h|$ to denote the pointwise absolute value of a harmonic function h and $|h|_{\mathcal{H}}$ for the lattice absolute value of h. If $h_f = I(f)$ and $h_g = I(g)$, then

$$h_f \vee h_g = I(f \vee g),\ \ h_f \wedge h_g = I(f \wedge g),\ \text{ and }\ |h_f|_{\mathcal{H}} = I(|f|).$$

We now set $\mathbf{D}_o := \mathcal{D}_o\text{-Lmd}\,\mathbf{D}$. An internal harmonic function $h \in \mathbf{D}$ is in $\mathbf{D}_o$ iff $|h|_{\mathcal{H}}$ has a limited maximum value on the extension of each standard compact set. If g is a positive function in $\mathbf{D}$ and $g(w_0)$ is limited at some standard $w_0 \in W$, then by Harnack's inequality, $g \in \mathbf{D}_o$. The mapping

$$w \mapsto {}^\circ g(w),\ w \in W$$

produces a standard harmonic function on W. Every $g \in \mathbf{D}_o$ is the difference of two nonnegative functions in $\mathbf{D}_o$. Of course $\mathbf{D}$ is not a standard space. Nevertheless, if $|g|_{\mathcal{H}}(w_0)$ is limited at some standard point w_0, we will write, $\mathrm{st}_{ucc}(g)$ for the standard harmonic function given by the map $w \mapsto {}^\circ g(w)$ on W.

The nonstandard hull $\hat{\mathbf{D}}_o$ is quite different from the space that would result from equivalence classes defined in terms of the ucc topology. For example, let P_0 be the Poisson kernel function at 1 on the unit disk and P_Θ be the nonstandard Poisson kernel function for the point $e^{i\Theta}$ where $\Theta \approx 0$ but Θ is not too close to 0. Then in $\hat{\mathbf{D}}_o$, $\overline{I({}^*P_0|X) - I(P_\Theta|X)}$ is not the zero function. Indeed this element of the nonstandard hull $\hat{\mathbf{D}}_o$ gives a reasonable representation of a dipole.

By transfer of the maximum principle, our integral I is $\mathbf{D}_o$-continuous. Since L_1 consists of real-valued functions on X, if $f \in L_1$ and φ is an internal representative of f, then $I(\varphi) \in \mathbf{D}_o$ by Proposition 3.3. The range of J is contained in the space $\hat{\mathbf{D}}_o$. We would like, however, to obtain ordinary harmonic functions from J.

What we will be able to obtain are the bounded and quasibounded harmonic functions on W. A nonnegative harmonic function is quasibounded if it is the monotone increasing limit of bounded harmonic functions. The difference of two nonnegative quasibounded harmonic functions is again called quasibounded. Of course, a bounded harmonic function is quasibounded. In terms of integral representations on, say, the Martin boundary, the quasibounded harmonic functions are the functions produced by L^1-densities (see, for example, [14]).

Let S be the mapping on $\hat{D}_o$ given at each $\bar{g} \in \hat{\mathbf{D}}_o$ by setting $S(\bar{g}) = \mathrm{st}_{ucc}(g)$. It is easy to see that S is a well-defined, positive linear mapping on $\hat{\mathbf{D}}_o$. Now for each $f \in L_1$, we set

$$J^{'}(f) = S \circ J(f).$$

If f is a nonnegative function in L_1 and for each $n \in \mathbb{N} \quad f_n := f \wedge n$, then $f = \sup_n f_n$. By Theorem 3.6 and Proposition 2.6,

$$J(f) = \sup \ J(f_n) = \lim \ J(f_n)$$

in $\hat{\mathbf{D}}_o$. Since we are working with nonnegative differences,

$$J^{'}(f) = \sup \ J^{'}(f_n) = \lim \ J^{'}(f_n);$$

that is, $J^{'}(f)$ is quasi-bounded. On the other hand, if h is a nonnegative, quasi-bounded harmonic function on W, then

$$h = \sup \ J^{'}\left({}^\circ\left((h|X) \wedge n\right)\right).$$

It follows that there is a real-valued $f \in L_1$ with $h = J^{'}(f)$. We have thus established the following variation of the integral representation for quasi-bounded harmonic functions with respect to the Martin boundary and the boundary developed by the first author in [13] and [14].

Theorem 6.1. *A harmonic function on W is quasi-bounded if and only if it equals $J^{'}(f)$ for some $f \in L_1$.*

7. The Bochner Integral

For our last example, we fix a standard normed vector lattice $(\mathbf{E}, +, \cdot, \|\cdot\|, \leq)$ with $\mathbf{E} \neq \{0\}$; the topology on $\mathbf{E}$ is derived from the norm. Initially, we assume that for all a, $b \in \mathbf{E}$, $|b| \leq a \Rightarrow \|b\| \leq \|a\|$, but if the open balls centered at 0 are semi-solid but not solid, we replace the norm of each element with the norm of its lattice absolute value. Given this replacement, the following implication will hold:

$$\forall a, b \in \mathbf{E},\ |b| \leq |a| \Rightarrow \|b\| \leq \|a\|.$$

That is, the norm is a Riesz norm and the space $(\mathbf{E}, +, \cdot, \|\cdot\|, \leq)$ is a normed Riesz space. Now, $\|b\| = \||b|\|$ for each $b \in \mathbf{E}$ and the open balls centered at 0 are all solid.

For this section, $(\mathbf{B}, +, \cdot, \|\cdot\|, \leq)$ is the nonstandard extension of $(\mathbf{E}, +, \cdot, \|\cdot\|, \leq)$. Note that we omit the stars in the notation for the extensions of the vector operations, the norm, and the ordering. The internal topology on $\mathbf{B}$ is derived from the internal norm. For each $\rho > 0$ in ${}^*\mathbb{R}$, we have an internal, solid, open ball $U_\rho(0) := \{a \in \mathbf{B} : \|a\| < \rho\}$.

The internal neighborhood system $\mathcal{U}$ of 0, and the external $\mathbb{R}$-neighborhood base $\mathcal{U}_o$ of 0 are chosen as in Example I of Section 2. We let $\mathbf{B}_o$ equal to the set of $\mathcal{U}_o$-limited elements of $\mathbf{B}$, and we write $\hat{\mathbf{E}}$ for $\hat{\mathbf{B}}_o$. We identify each element a in $\mathbf{E}$ with the equivalence class containing ${}^*a$ in $\hat{\mathbf{E}}$, and thus consider $\mathbf{E}$ to be a subset of $\hat{\mathbf{E}}$.

For the operations, norm, and ordering on $\hat{\mathbf{E}}$, we write $+, \cdot, \|\cdot\|$, and $\leq$. The norm of each $\bar{a} \in \hat{\mathbf{E}}$ is the standard part of $\|a\|$. By the saturation assumption, $\hat{\mathbf{E}}$ is complete with respect to the metric generated by this norm ([18], Theorem 3.16.1).

Fix an internal measure space $(X, \mathcal{A}, \mu)$ with ${}^\circ\mu(X) < +\infty$, and let $(X, L_\mu(\mathcal{A}), \hat{\mu})$ be the corresponding Loeb space. We let L be the space of internal $\mathcal{A}$-simple functions with values in $\mathbf{B}$. That is, each $\varphi \in L$ takes only a finite or hyperfinite number of values in $\mathbf{B}$, and for each $a \in \mathbf{B}$, $\{x \in X : \varphi(x) = a\} \in \mathcal{A}$. As usual, the internal L_1- norm $\|\cdot\|_1$ on L is given by

$$\|\varphi\|_1 = \int_X \|\varphi\| \, d\mu.$$

Fix $\varphi \in L$. The function $\|\varphi\|$ is **S-integrable**, if for all $\gamma \in {}^*N_\infty$, $\int_{\{\|\varphi\| \geq \gamma\}} \|\varphi\| \, d\mu \approx 0$. Recall that $\|\varphi\|$ is S-integrable, iff ${}^\circ \int_X \|\varphi\| \, d\mu < \infty$ and for all $A \in \mathcal{A}$ with $\mu(\mathcal{A}) \approx 0$, ${}^\circ \int_A \|\varphi\| \, d\mu = 0$, in which case $\int_X {}^\circ \|\varphi\| \, d\hat{\mu} = {}^\circ \int_X \|\varphi\| \, d\mu$. Extending the terminology of [2], we will say a function $\varphi \in L$ is **SL-integrable**, if $\|\varphi\|$ is S-integrable.

The outer measure $\hat{\mu}^{\text{out}}$ is defined for all subsets $B \subset X$ by

$$\hat{\mu}^{\text{out}}(B) := \inf \{{}^\circ\mu(A) \mid B \subset A \in \mathcal{A}\}.$$

Of course $\hat{\mu}^{\text{out}}(B) = \hat{\mu}(B)$ if $B \in L_\mu(\mathcal{A})$, and in particular, if $\hat{\mu}^{\text{out}}(B) = 0$.

An $\hat{\mathbf{E}}$-valued function f defined on X is called $L_\mu(\mathcal{A})$-**measurable**, if there is a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ of $L_\mu(\mathcal{A})$-simple functions **converging to f in measure**; i.e., for all $\varepsilon > 0$ in $\mathbb{R}$

$$\lim_{n \to \infty} \hat{\mu}^{\text{out}} \{\|g_n - f\| \geq \varepsilon\} = 0.$$

A function $\varphi \in L$ is called a μ-**lifting** of a function $f : X \to \hat{\mathbf{E}}$, if for $\hat{\mu}$ -almost all $x \in X$, $\varphi(x) \in \mathbf{B}_o$ and $f(x) = \overline{\varphi(x)}$.

Proposition 7.1. *(1) Every $\hat{\mathbf{E}}$-valued function f on X which is $L_\mu(\mathcal{A})$-measurable has a μ-lifting $\varphi \in L$. We may assume φ is SL-integrable, if $\int_X \|f\| \, d\hat{\mu} < \infty$.*
(2) If f is just an $\mathbf{E}$-valued function on X and f has a μ-lifting $\varphi \in L$, then f is $L_\mu(\mathcal{A})$-measurable. Indeed, in this case, there is a sequence of $L_\mu(\mathcal{A})$-simple functions converging $\hat{\mu}$-a.e. to f.

Proof. (1) Assume $f : X \to \hat{\mathbf{E}}$ is $L_\mu(\mathcal{A})$-measurable. Fix a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ of $L_\mu(\mathcal{A})$-simple functions g_n converging to f in measure and a standard function $h : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all k, $m \geq h(n)$

$$\hat{\mu}^{\text{out}} \left\{ \|g_k - f\| \geq \frac{1}{2n} \right\} < \frac{1}{2n}, \text{ whence } \hat{\mu} \left\{ \|g_k - g_m\| \geq \frac{1}{n} \right\} < \frac{1}{n}.$$

For all $B \in L_\mu(\mathcal{A})$ there is an $A \in \mathcal{A}$ with $\hat{\mu}(B \triangle A) = 0$. Moreover, each g_k takes values which may all be obtained from a finite set S in $\mathbf{B}_o$. Therefore, there is a $\varphi_k \in L$, with range contained in S such that $\overline{\varphi_k} = g_k$ $\hat{\mu}$-a.e. Now for all $k, m \geq h(n)$

$$\mu \left\{ \|\varphi_k - \varphi_m\| \geq \frac{1}{n} \right\} \approx \hat{\mu} \left\{ \|\varphi_k - \varphi_m\| \geq \frac{1}{n} \right\} \leq \hat{\mu} \left\{ \|g_k - g_m\| \geq \frac{1}{n} \right\} < \frac{1}{n}.$$

By saturation, there exists a $\varphi \in L$ such that for all $n \in \mathbb{N}$

$$\mu \left\{ \left\|\varphi - \varphi_{h(n)}\right\| \geq \frac{1}{n} \right\} < \frac{1}{n}, \text{ whence } \hat{\mu} \left\{ \left\|\overline{\varphi} - \overline{\varphi_{h(n)}}\right\| \geq \frac{3}{2n} \right\} \leq \frac{1}{n}.$$

Since ${}^\circ \left\|\varphi_{h(n)}\right\| < \infty$ for all $n \in \mathbb{N}$, ${}^\circ \|\varphi\| < \infty$ $\hat{\mu}$-a.e. Therefore, $\varphi \in \mathbf{B}_o$ $\hat{\mu}$-a.e., and

$$\hat{\mu}^{\text{out}} \left\{ \|\overline{\varphi} - f\| \geq \frac{2}{n} \right\} \leq \hat{\mu} \left\{ \left\|\overline{\varphi} - \overline{\varphi_{h(n)}}\right\| \geq \frac{3}{2n} \right\} + \hat{\mu}^{\text{out}} \left\{ \left\|\overline{\varphi_{h(n)}} - f\right\| \geq \frac{1}{2n} \right\} < \frac{2}{n}$$

for all $n \in \mathbb{N}$. It follows that $\hat{\mu}^{\text{out}} \{\|\overline{\varphi} - f\| \neq 0\} = 0$, so $\overline{\varphi} = f$ $\hat{\mu}$-a.e; that is, φ is a μ-lifting of f. Suppose in addition that $\int_X \|f\| \, d\hat{\mu} < \infty$. Since $\overline{\varphi} = f$ $\hat{\mu}$-a.e., we may replace φ with 0 on a null set where $\|\varphi\|$ is larger than some small unlimited integer and thus obtain an SL-integrable μ-lifting of f.

(2) Assume now that f is $\mathbf{E}$-valued and has a μ-lifting $\varphi \in L$. Since $\varphi \approx f$ $\hat{\mu}$-a.e., there exists a decreasing sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that for each $n \in \mathbb{N}$, $\mu(A_n) < 1/n$ and $\varphi(x) \approx f(x)$ for all $x \in X \setminus A_n$. Fix $n \in \mathbb{N}$. Since $\{\varphi(x) : x \in X \setminus A_n\}$ is an internal set of nearstandard points of ${}^*\mathbf{E}$, by Luxemburg's Theorem 3.1.6. in [18],

$$K_n = \{f(x) : x \in X \setminus A_n\} = \{{}^\circ\varphi(x) : x \in X \setminus A_n\}$$

is compact in $\mathbf{E}$. We fix a covering of K_n by a finite collection of standard open balls O_i, $1 \leq i \leq m$, each having radius $r_i < 1/2n$

and center $y_i \in \mathbf{E}$. Since $\{\varphi(x) : x \in X \setminus A_n\} \subset \bigcup_{i=1}^{m} {}^*O_i$, we may construct simple functions f_n on X as follows: For each $x \in A_n$ set $f_n(x) = 0$. For each $x \notin A_n$, set $f_n(x) = y_i$ if $\varphi(x) \in {}^*O_i$ and $\varphi(x) \notin {}^*O_1 \cup ... \cup {}^*O_{i-1}$, $1 \le i \le m$. We thus obtain for all $x \in X \setminus A_n$,

$$\|f(x) - f_n(x)\| \le \|f(x) - {}^\circ\varphi(x)\| + \|{}^\circ\varphi(x) - f_n(x)\| < \frac{1}{n}.$$

It follows that f_n converges to f pointwise on $X \setminus \cap A_n$. □

In her 1994 dissertation and related article [28], B. Zimmer has constructed examples of $\hat{\mathbf{E}}$-valued functions which are not $L_\mu(\mathcal{A})$-measurable but which have μ-liftings in L. The spaces used for these examples are the ones discussed here in Section 5.

Following standard definitions, an $\hat{\mathbf{E}}$-valued function f on X is called **Bochner integrable**, if there exists a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of $L_\mu(\mathcal{A})$-simple functions f_n converging to f in measure with

$$\lim_{n,m\to\infty} \int_X \|f_n - f_m\| \, d\hat{\mu} = 0.$$

In this case

$$B\text{-}\int_X f \, d\hat{\mu} := \lim_{n\to\infty} \int_X f_n \, d\hat{\mu}$$

is called the **Bochner Integral** of f . We are employing here the obvious meaning for the integral of $L_\mu(\mathcal{A})$-simple functions. We will use the analogous meaning for internal integrals of functions in L with respect to μ.

Our next theorem extends lifting theorems in [12] and [2] to Bochner integrable functions. For $\mathbf{E}$-valued functions, where $\mathbf{E}$ is complete, it extends a result on the Bochner Integral of Capinsky and Cutland ([6], Proposition 3.2). We will use the notation ${}^\circ a$ for the standard part in $\mathbf{E}$ of a nearstandard point a in $\mathbf{B} = {}^*\mathbf{E}$.

Theorem 7.2. *Every $\hat{\mathbf{E}}$-valued Bochner integrable function f on X has an SL-integrable μ-lifting in L. For any such lifting φ , $\int_X \varphi \, d\mu \in \mathbf{B}_o$, and*

$$B\text{-}\int_X f \, d\hat{\mu} = \overline{\int_X \varphi \, d\mu}.$$

On the other hand, if $\mathbf{E}$ is complete and f is an $\mathbf{E}$-valued function on X, then f has an SL-integrable μ-lifting $\varphi \in L$, iff f is Bochner integrable. In this case, for any such lifting φ, $\int_X \varphi \, d\mu$ is near-standard *and*

$$B\text{-}\int_X f\, d\hat{\mu} = {}^\circ \int_X \varphi\, d\mu.$$

Proof. Fix a Bochner integrable $f : X \to \hat{\mathbf{E}}$. Since f is $L_\mu(\mathcal{A})$-measurable, we may fix sequences $\langle g_n \rangle_{n\in\mathbb{N}}$ and $\langle \varphi_n \rangle_{n\in\mathbb{N}}$ and a function $h : \mathbb{N} \to \mathbb{N}$ as in the proof of Part 1 in Proposition 7.1; we may further assume that

$$\lim_{n\to\infty} \int_X \|g_n - g_m\|\, d\hat{\mu} = 0. \tag{7.1}$$

Recall that for all $k, m \geq h(n)$

$$\mu\left\{\|\varphi_k - \varphi_m\| \geq \frac{1}{n}\right\} < \frac{1}{n}. \tag{7.2}$$

Since $\overline{\varphi_k} = g_k$ $\hat{\mu}$-a.e., and the range of φ_k is a finite set in $\mathbf{B}_o$,

$$\int_X g_k\, d\hat{\mu} = \overline{\int \varphi_k\, d\mu}. \tag{7.3}$$

We may assume that h has been chosen so that for all $k, m \geq h(n)$

$$\int_X \|\varphi_k - \varphi_m\|\, d\mu < \frac{1}{n}. \tag{7.4}$$

By Formulae 7.2 and 7.4 and saturation, there exists $\varphi \in L$ such that for all $n \in \mathbb{N}$

$$\mu\left\{\left\|\varphi - \varphi_{h(n)}\right\| \geq \frac{1}{n}\right\} < \frac{1}{n}, \text{ and } \int_X \left\|\varphi - \varphi_{h(n)}\right\| d\mu < \frac{1}{n}. \tag{7.5}$$

As in the proof of Part 1 of Proposition 7.1, φ is a μ-lifting of f. Moreover, φ is SL-integrable, because for all $A \in \mathcal{A}$ with $\mu(A) \approx 0$ and all $n \in \mathbb{N}$,

$$\int_A \|\varphi\|\, d\mu \leq \int_A \left\|\varphi - \varphi_{h(n)}\right\|\, d\mu + \int_A \left\|\varphi_{h(n)}\right\|\, d\mu < \frac{2}{n},$$

and by a similar calculation, $\int_X \|\varphi\|\, d\mu$ is limited. From Formulae 7.3 and 7.5 it now follows that

$$B\text{-}\int_X f\, d\hat{\mu} = \lim_{k\to\infty} \int g_k\, d\hat{\mu} = \lim_{k\to\infty} \overline{\int \varphi_k\, d\mu} = \overline{\int \varphi\, d\mu}. \tag{7.6}$$

If we assume that $\mathbf{E}$ is complete and f is actually an $\mathbf{E}$-valued function on X, then we may assume the functions g_n are also $\mathbf{E}$-valued. In this case, Equation 7.6 has the form

$$B\text{-}\int_X f\, d\hat{\mu} = \lim_{k\to\infty} \int g_k\, d\hat{\mu} = \lim_{k\to\infty} {}^\circ \int_X \varphi_k\, d\mu = {}^\circ \int \varphi\, d\mu.$$

If ψ is another SL-integrable μ-lifting of f ,then, since $\int_X \|\varphi - \psi\| \, d\mu \approx 0$,

$$^\circ\int_X \psi \, d\mu = {}^\circ\int_X \varphi \, d\mu = B\text{-}\int_X f \, d\hat{\mu}.$$

Now assume that f is a mapping from X into $\mathbf{E}$ and that f has an SL-integrable μ-lifting $\varphi \in L$. By Proposition 7.1, f is $L_\mu(\mathcal{A})$-measurable and there exists a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of $L_\mu(\mathcal{A})$ -simple functions $f_n : X \to \mathbf{E}$ converging to f $\hat{\mu}$-a.e. Again there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that for all $k \geq h(n)$, $\hat{\mu}\{\|f_k - f\| \geq 1/n\} < 1/n$. Moreover, $\int_X \|f\| \, d\hat{\mu} < \infty$, because $\|\varphi\|$ is S-integrable and $^\circ\varphi = f$ $\hat{\mu}$-a.e. We will show that $\lim_{n\to\infty} \int_X \|f_n - f\| \, d\hat{\mu} = 0$. Fix $\varepsilon > 0$ in $\mathbb{R}$. Because $\|f_n\|$ and $\|f\|$ are $\hat{\mu}$-integrable, $\|f - f_n\|$ is $\hat{\mu}$-integrable. Therefore there exists $\delta > 0$ in $\mathbb{R}$ such that $\int_A \|f_n - f\| \, d\hat{\mu} < \varepsilon/2$ for all $A \in L_\mu(\mathcal{A})$ with $\hat{\mu}(A) < \delta$. Choose $n_o \in \mathbb{N}$ with $1/n_0 < \delta$ and $\hat{\mu}(X)/n_0 < \varepsilon/2$. Because $\hat{\mu}\{\|f_k - f\| \geq 1/n_0\} < \delta$, for all $k \geq h(n_o)$, we have

$$\begin{aligned}
\int_X \|f_k - f\| \, d\hat{\mu} &= \int_{\{\|f_k - f\| \geq 1/n_0\}} \|f_k - f\| \, d\hat{\mu} + \int_{\{\|f_k - f\| < 1/n_0\}} \|f_k - f\| \, d\hat{\mu} \\
&< \frac{\varepsilon}{2} + \frac{1}{n_o} \cdot \hat{\mu}(X) < \varepsilon.
\end{aligned}$$

It follows that $\lim_{n,m\to\infty} \int_X \|f_n - f_m\| \, d\hat{\mu} = 0$, whence f is Bochner integrable. □

Theorem 7.2 can be used to establish the well-known fact that if $f : X \to \mathbf{E}$, where $\mathbf{E}$ is complete, then f is Bochner integrable if and only if f is $L_\mu(\mathcal{A})$-measurable and $\int_X \|f\| \, d\hat{\mu} < \infty$. We next fit the Bochner Integral into our general setting and show that in general our general integral is an extension of the Bochner Integral.

For this section, we set the space $\mathbf{D}$ of our general theory equal to L, the space of simple functions employed above. In $\mathbf{D}$, we identify two functions φ and ψ if $\varphi(x) = \psi(x)$ for μ-almost all $x \in X$. With this identification, $\mathbf{D}$ is a vector lattice, where for all functions φ, $\psi \in \mathbf{D}$, $\varphi \leq \psi \Leftrightarrow \varphi(x) \leq \psi(x)$ for μ-almost all $x \in X$.

Departing from the structure of L, we make $\mathbf{D}$ a normed vector lattice by setting $\|\varphi\|_1 := \int_X \|\varphi\| \, d\mu$ for each $\varphi \in \mathbf{D}$. As usual, the neighborhood system $\mathcal{D}$ at 0 is be the set of internal open balls centered at 0 with positive radius $\rho \in {}^*\mathbb{R}$. We let $\mathcal{D}_o$ be the subset of $\mathcal{D}$ consisting of balls of radius $1/n$, $n \in \mathbb{N}$. If φ and ψ are in L and $|\varphi| \leq |\psi|$ μ-a.e., then $\|\varphi\|_1 \leq \|\psi\|_1$, so the elements of $\mathcal{D}$ are solid.

For each $\varphi \in L$, we let $I(\varphi)$ denote the corresponding equivalence class with respect to μ-a.e. equality. The mapping $I : L \to \mathbf{D}$ is the internal integral of our general theory. We will show by example that this map, and not the map $\varphi \mapsto \int_X \varphi d\mu$, is the right choice of I for extending the Bochner Integral. From I, we construct spaces L_0 and L_1 as well as an external integral J using our general theory. The next result shows that the SL-integrable functions in L are exactly the internal representatives of functions in L_1.

Theorem 7.3. *If $\overline{\varphi + h} \in L_1$, where $\varphi \in L$, $h \in L_0$ and $\varphi + h \in \mathbf{B}_o$ pointwise, then φ is an SL-integrable, μ-lifting of $f = \overline{\varphi + h}$. Conversely, if $\varphi : X \to {}^*\mathbf{E}$ is SL-integrable, so $\varphi(x) \in \mathbf{B}_o$ for $\hat{\mu}$-almost all $x \in X$, and if we set $h(x) := 0$ when $\varphi(x) \in \mathbf{B}_o$ while $h(x) := -\varphi(x)$ when $\varphi(x) \notin \mathbf{B}_o$, then $f = \overline{\varphi + h} \in L_1$.*

Proof. Assume $f = \overline{\varphi + h} \in L_1$. We want to establish the SL-integrability of φ. Given $\gamma \in {}^*\mathbb{N}_\infty$, $A := \{x \in X : \gamma \leq \|\varphi(x)\|\}$, and $\varepsilon > 0$ in $\mathbb{R}$, we need only show that $\int_A \|\varphi\| \, d\mu < \varepsilon$. Since $h \in L_0$, there exists a $\psi \in L$ with $|h| \leq \psi$ and $\|I(\psi)\|_1 = \int_X \|\psi\| \, d\mu < \varepsilon/2$. Fix $x \in A$. Since $\gamma \leq \|\varphi(x)\|$ and $\|\varphi(x) + h(x)\|$ is limited in ${}^*\mathbb{R}$, $\|h(x)\|$ is unlimited in ${}^*\mathbb{R}$, so

$$\|\varphi(x)\| \leq \|h(x)\| + \|\varphi(x) + h(x)\| \leq 2\,\|h(x)\| \leq 2\,\|\psi(x)\|\,.$$

Therefore, $\int_A \|\varphi\| \, d\mu \leq 2 \int_A \|\psi\| \, d\mu < \varepsilon$.

To show that $\overline{\varphi} = f$ $\hat{\mu}$-a.e., we fix $n \in \mathbb{N}$ and $\varepsilon > 0$ in $\mathbb{R}$ and show that

$$\widehat{\mu}^{\text{out}} \left\{ x : \|h(x)\| \geq \frac{1}{n} \right\} \leq \varepsilon.$$

This follows from the existence of a $\psi \in L$ with $|h| \leq \psi$ and $\int_X \|\psi\| \, d\mu < \varepsilon/n$ since

$$\widehat{\mu}^{\text{out}} \{\|h\| \geq 1/n\} \lessapprox \mu \left\{ \|\psi\| \geq \frac{1}{n} \right\} \leq n \cdot \int_X \|\psi\| \, d\mu < \varepsilon.$$

Now fix an SL-integrable φ. Set $h(x) = 0$ when $\varphi(x) \in \mathbf{B}_o$ and $h(x) = -\varphi(x)$ when $\varphi(x) \notin \mathbf{B}_o$, so $\varphi(x) + h(x) \in \mathbf{B}_0$ for all $x \in X$. We must show that $h \in L_0$. Given $\varepsilon > 0$ in $\mathbb{R}$, by the S-integrability of $\|\varphi\|$ there is an $m \in \mathbb{N}$ such that $\int_{\{\|\varphi\| \geq m\}} \|\varphi\| \, d\mu < \varepsilon$. Let $A = \{\|\varphi(x)\| \geq m\}$, and set $\psi(x) := |\varphi(x)| \cdot 1_A$. Then $\psi \in L$, $|h| \leq \psi$, and

$$\int_X \|\psi\| \, d\mu = \int_A \|\psi\| \, d\mu = \int_A \|\varphi\| \, d\mu < \varepsilon.$$

It follows that $h \in L_0$. $\square$

Next, we compare L_1 with the space of Bochner integrable functions. We identify two Bochner integrable functions f and g if they are equal

$\hat{\mu}$-a.e., and we write $L^B(\hat{\mu})$ to denote the corresponding space of equivalence classes. We can also identify two functions f and g in our external space L_1 if they are equal $\hat{\mu}$-a.e. To indicate this identification, we will write $L_1(\hat{\mu})$ instead of L_1. We give $L_1(\hat{\mu})$ the vector operations and ordering inherited from L_1; these are well-defined. For example, when f and g are in L_1, we have $f \leq g$ in $L_1(\hat{\mu})$ if pointwise $f \leq g$ $\widehat{\mu}$-a.e. Since in L_1, $\int_X \|f\| d\hat{\mu} = 0$ if and only if $f = 0$ $\hat{\mu}$-a.e. on X, setting $\|f\|_1 := \int_X \|f\| d\hat{\mu}$ puts a well-defined norm on $L_1(\hat{\mu})$.

Proposition 7.4. *In general, $L^B(\hat{\mu}) \subseteq L_1(\hat{\mu})$. We have equality if the dimension of $\mathbf{E}$ is finite, i.e., if $\mathbf{E} = \hat{\mathbf{E}}$.*

Proof. Assume $f \in L^B(\hat{\mu})$. By Part 1 of Proposition 7.1, the function f has an SL-integrable μ-lifting φ. By Theorem 7.3, there exists $h \in L_0$ such that $\overline{\varphi + h} \in L_1$ and $\overline{\varphi} = \overline{\varphi + h}$ $\hat{\mu}$-a.e. Since $f = \overline{\varphi + h}$ $\hat{\mu}$-a.e., we may identify f and $\overline{\varphi + h}$, thus $f \in L_1(\hat{\mu})$. Now assume that the dimension of $\mathbf{E}$ is finite, and fix $f = {}^\circ(\varphi + h) \in L_1(\hat{\mu})$. By Theorem 7.3, φ is an SL-integrable μ-lifting of f, so $f = {}^\circ\varphi$ $\hat{\mu}$-a.e. By Part 2 of Proposition 7.1, f is $L_\mu(\mathcal{A})$-measurable. Since $\|\varphi\| \approx \|f\|$ $\hat{\mu}$-a.e. and $\|\varphi\|$ is S-integrable, $\int_X \|f\| \, d\hat{\mu} < \infty$, whence $f \in L^B(\hat{\mu})$. □

Theorem 7.5. *The space $L_1(\hat{\mu})$ is a Banach lattice.*

Proof. We need only establish the convergence of an arbitrary Cauchy sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $L_1(\hat{\mu})$. For each $n \in \mathbb{N}$, we may choose an internal representative $\psi_n \in L$ of f_n and an $h(n) \in \mathbb{N}$ such that for all $m, k \geq h(n)$, $\int_X \|\psi_k - \psi_m\| \, d\mu < 1/n$. By saturation, there exists a $\psi \in L$ such that for all $n \in \mathbb{N}$

$$\int_X \left\|\psi - \psi_{h(n)}\right\| d\mu < \frac{1}{n}.$$

As in the proof of Theorem 7.2, one may show that ψ is SL-integrable. By Theorem 7.3 ψ is an internal representative of a function $f \in L_1(\hat{\mu})$ which is the desired limit since

$$0 = \lim_{n\to\infty} {}^\circ \int \|\psi - \psi_n\| \, d\mu = \lim_{n\to\infty} \int \|f - f_n\| \, d\hat{\mu}. \ \square$$

We now single out the interval closed, $\mathbb{R}$-sublattice $\mathbf{D}_o$ of $\mathbf{D}$ consisting of those (internal) equivalence classes of functions in L for which the representative functions are SL-integrable on X. It follows from Theorem 7.3 that for this choice of $\mathbf{D}_o$, $L_1 = L_1^\circ$. It also follows from Theorem 7.3 that for any $\varphi \in L$, the equivalence class $[\varphi]$ containing φ is in $\mathbf{D}_o$ if and only if φ is an internal representative of some $f \in L_1$. In this case, since $I(\varphi) := [\varphi]$, $J(f) = \overline{[\varphi]}$ in $\hat{\mathbf{D}}_o$. We will also use J to denote the corresponding map defined on $L_1(\hat{\mu})$.

Theorem 7.6. *The mapping J is a well-defined, order-preserving, linear isometry from $L_1(\hat{\mu})$ onto $\hat{\mathbf{D}}_o$.*

Proof. Fix arbitrary elements f and g in L_1 and corresponding SL-integrable representatives φ_f and φ_g in L. Note first that if $f = g$ $\hat{\mu}$-a.e., then $\overline{\varphi_f}(x) = \overline{\varphi_g}(x)$ in $\hat{\mathbf{E}}$ for $\hat{\mu}$-a.e. $x \in X$. It follows that $\int_X \|\varphi_f - \varphi_g\| d\mu \approx 0$, whence $\left\|\varphi_f - \varphi_g\right\|_1 \approx 0$. By definition, $\overline{[\varphi_f]} = \overline{[\varphi_g]}$ in $\hat{\mathbf{D}}_o$. Thus we have shown that J is well-defined. Clearly, J is a linear map. To show it is order preserving, note that

$$\begin{aligned} f \quad &\leq \quad g \text{ in } L_1(\hat{\mu}) \Leftrightarrow \varphi_f \mathrel{\widetilde{\leq}} \varphi_g \ \hat{\mu}\text{-a.e.} \Leftrightarrow (\varphi_f \vee \varphi_g) - \varphi_g \approx 0 \ \hat{\mu}\text{-a.e.} \\ &\Leftrightarrow \quad \int_X \left\|(\varphi_f \vee \varphi_g) - \varphi_g\right\| \, d\mu \approx 0 \Leftrightarrow \overline{[\varphi_f]} \mathrel{\widehat{\leq}} \overline{[\varphi_g]} \text{ in } \hat{\mathbf{D}}_o. \end{aligned}$$

To show that J is an isometry, note that $\left\|\overline{[\varphi_f]}\right\|_1 = {}^\circ\|\varphi_f\|_1 = \|f\|_1$. By Theorem 7.3, each SL-integrable $\varphi \in L$ represents some $f \in L_1$, so the range of J is all of $\hat{\mathbf{D}}_o$. □

Corollary 7.7. *The space $\hat{\mathbf{D}}_o$ is a Banach lattice.*

For each $\varphi \in L$, set $I'(\varphi) := \int_X \varphi \, d\mu$ in ${}^*\mathbf{E}$. Let L_0' denote the space of null functions on X constructed using the integral I'. It would seem at first glance that I' would be the right internal integral to use in formulating the theory of Bochner integration. Recall, however, that we want two functions on X which differ by a null function to be essentially the same. In particular, we want Theorem 7.3 to hold. The following example shows that this property fails for functions in L_0'.

Example. Let $\mathbf{E} = \ell^2$ and $X = {}^*\mathbb{N}$. Let $\mathcal{A}$ be the collection of internal subsets of ${}^*\mathbb{N}$. As in Section 5, fix an unlimited $H \in {}^*\mathbb{N}$, and let μ be the internal measure on ${}^*\mathbb{N}$ given by setting $\mu(i) = 1/H$ for $1 \leq i \leq H$ and $\mu(i) = 0$ for $i > H$. Let δ^i denote the internal, ${}^*\mathbb{R}$-valued sequence which is identically equal to 0 except at i where it is equal to 1. Define $\varphi \in L$ as follows: For $i \leq H$, $\varphi(i) = \delta^i$ and for $i > H$, $\varphi(i) = 0$. Then $\varphi \in L_0'$ but $\varphi \notin L_0$, since in $\mathbf{B} = {}^*\ell^2$,

$$\|I'(\varphi)\| = \|\Sigma_{i=1}^H \frac{1}{H} \cdot \delta^i\| = [\Sigma_{i=1}^H \frac{1}{H^2}]^{1/2} = H^{-1/2} \approx 0,$$

while in $\mathbf{D}$

$$\|I(\varphi)\|_1 = \|\varphi\|_1 = \Sigma_{i=1}^H (1/H) = 1.$$

Notice that in $\mathbf{B}$, $\|\varphi(i)\| = 1$ for each $i \leq H$.

We have used the mapping I to form the class of functions on which we can apply an extended Bochner Integral. Having done this, the mapping I' is now used to construct that extended integral. First, we define a positive linear mapping $T : \hat{\mathbf{D}}_o \to \hat{\mathbf{E}}$ by setting $T\left(\overline{[\varphi]}\right) := \overline{I'(\varphi)}$ for each $\varphi \in \mathbf{D}_o$. It is easy to see that T is well-defined on $\hat{\mathbf{D}}_o$. Our last two results show that the integral given by $T \circ J$ extends the Bochner Integral and satisfies a monotone convergence property.

Theorem 7.8. *The positive linear mapping $T \circ J : L_1(\hat{\mu}) \to \hat{\mathbf{E}}$ is an extension of the Bochner Integral which maps $L^B(\hat{\mu})$ into $\hat{\mathbf{E}}$. If $\mathbf{E}$ is complete and f is $\mathbf{E}$-valued in $L_1(\hat{\mu})$, the value of $T \circ J(f)$ in $\mathbf{E} \subseteq \hat{\mathbf{E}}$ is the Bochner Integral of f.*

Proof. The result follows from Theorem 7.2, since the value of $T \circ J$ at each $f \in L_1(\hat{\mu})$ is defined in terms of any internal representative φ_f of f by

$$T \circ J(f) = \overline{I'(\varphi_f)} = \overline{\int_X \varphi_f \, d\mu} \in \hat{\mathbf{E}}. \ \square$$

Theorem 7.9 (Monotone Convergence)**.** *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be an increasing sequence in $L_1(\hat{\mu})$ such that pointwise for $\hat{\mu}$-a.e. $x \in X$, $f(x) := \sup_n f_n(x)$ exists in $\hat{\mathbf{E}}$ and $\sup_n \int_X \|f_n\| \, d\hat{\mu} < \infty$. Then $f \in L_1(\hat{\mu})$ and $T \circ J(f) = \lim_n T \circ J(f_n)$.*

Proof. By real-valued convergence theorems, we have $\lim_{n,m\to\infty} \int_X \|f_n - f_m\| \, d\hat{\mu} = 0$. The result now follows from Proposition 2.6, Theorem 3.6, and Corollary 7.7. $\square$

Remark. In her 1994 dissertation and related article [28], Beate Zimmer has extended the results of this section to general Banach spaces. Among her results is a dominated convergence theorem for the extended integral. A similar theorem for functions with uniformly integrable norms is also valid. An important aspect of Zimmer's work for both the lattice setting and the general setting relates to whether or not $\hat{\mathbf{E}}$ has the Radon-Nikodym Property. If it does, then $T \circ J$ is essentially the Pettis Integral restricted to $L_1(\hat{\mu})$. Moreover, in this case, every $f \in L_1(\hat{\mu})$ has the form $f = g + k$ where $g \in L^B(\hat{\mu})$ and the integral of $k \cdot 1_B$ is zero for each $B \in L_\mu(\mathcal{A})$. The extension of the Bochner Integral is more interesting, therefore, in the case that $\hat{\mathbf{E}}$ does not have the Radon-Nikodym Property, since in this case, the extension of the Bochner Integral is nontrivial.

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