

COVERING THEOREMS AND LEBESGUE INTEGRATION

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ABSTRACT. This paper shows how the Lebesgue integral can be obtained as a Riemann sum and provides an extension of the Morse Covering Theorem to open sets. Let X be a finite dimensional normed space; let μ be a Radon measure on X and let $\Omega \subseteq X$ be a μ -measurable set. For $\lambda \geq 1$, a μ -measurable set $S_\lambda(a) \subseteq X$ is a λ -Morse set with tag $a \in S_\lambda(a)$ if there is $r > 0$ such that $B(a, r) \subseteq S_\lambda(a) \subseteq B(a, \lambda r)$ and $S_\lambda(a)$ is starlike with respect to all points in the closed ball $B(a, r)$. Given a gauge $\delta : \Omega \rightarrow (0, 1]$ we say $S_\lambda(a)$ is δ -fine if $B(a, \lambda r) \subseteq B(a, \delta(a))$. If $f \geq 0$ is a μ -measurable function on Ω then $\int_\Omega f d\mu = F \in \mathbb{R}$ if and only if for some $\lambda \geq 1$ and all $\varepsilon > 0$ there is a gauge function δ so that $|\sum_n f(x_n) \mu(S(x_n)) - F| < \varepsilon$ for all sequences of disjoint λ -Morse sets that are δ -fine and cover all but a μ -null subset of Ω . This procedure can be applied separately to the positive and negative parts of a real-valued function on Ω . The covering condition $\mu(\Omega \setminus \cup_n S(x_n)) = 0$ can be satisfied due to the Morse Covering Theorem. The improved version given here says that for a fixed $\lambda \geq 1$, if A is the set of centers of a family of λ -Morse sets then A can be covered with the interiors of sets from at most κ pairwise disjoint subfamilies of the original family; an estimate for κ is given in terms of λ , X and its norm.

1 Introduction An attractive feature of the Riemann and Henstock integrals is that they can be defined in terms of Riemann sums. Suppose we wish to integrate a real-valued function f over a set Ω with respect to a measure μ . If we have disjoint measurable sets $\Omega_1, \dots, \Omega_N$ with union Ω (i.e., a partition of Ω), then we may try to define an integral as the limit of sums $\sum_{i=1}^N f(z_i) \mu(\Omega_i)$ for appropriate points $z_i \in \Omega$. One would hope that taking the sets Ω_i small enough and N large enough would make these sums close to the same value, which we then define to be the integral $\int_\Omega f d\mu$. When $\Omega \subseteq \mathbb{R}^d$, this is done in the Riemann case for Lebesgue measure and a bounded function f and bounded set Ω by choosing for the partition sets Ω_i uniformly small cubes and then choosing arbitrary points $z_i \in \Omega_i$. With the Henstock integral, f and Ω need no longer be bounded. For this case, the sets Ω_i are intervals satisfying a gauge condition. This means to begin with that we have a function $\delta : \Omega \rightarrow (0, R)$ for some positive R ; the mapping is called a **gauge function**, and we say the pair (z_i, Ω_i) is δ -**fine** if $z_i \in \Omega_i$ and Ω_i is contained in the closed ball with center z_i and radius $\delta(z_i)$. (When Ω is unbounded, the partition need only cover $\Omega \cap B(0, R)$ where $B(0, R)$ is a ball with center at the origin and large enough radius R determined by δ .) We obtain the McShane integral by dropping the restriction that $z_i \in \Omega_i$. See [8] for a discussion of these integrals.

All of these integration schemes revolve around finding a partition of Ω , which of course requires rather specialized sets Ω_i . Breaking this pattern, the Vitali Covering Theorem was used in [10] to define the Lebesgue integral with respect to Lebesgue measure on a finite

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interval of the real line. There the idea, given any $\eta > 0$, is to use a finite number of intervals so that $\lambda([a, b] \setminus \cup_{i=1}^N I_i) < \eta$. Here, we too will apply covering theory, but now to a measurable set Ω in a finite dimensional normed space X . We will obtain the Lebesgue integral with respect to a Radon measure as a series $\sum f(z_i)\mu(\Omega_i)$, where the sets $\Omega_i \subseteq X$ are disjoint and cover all but a null set of Ω . The sets Ω_i will be made small with respect to a gauge function. They can be balls or starlike sets (described in Section 2 below). It is the Besicovitch Covering Theorem for balls and the Morse Covering Theorem for starlike sets that enables us to fulfill the condition $\mu(\Omega \setminus \cup_i \Omega_i) = 0$. For this it is essential that we are working in a finite dimensional normed space, not just a metric space. These covering results are discussed below, and simplified proofs of strengthened versions are provided. We also note that by omitting a small part of the overall sum $\sum f(z_i)\mu(\Omega_i)$, we are able to restrict the points z_i to the set of points of approximate continuity of f , defined in terms of Morse covers in Section 3.

In the theory of Henstock and McShane integration, the appearance of the gauge function is rather mysterious: For all $\varepsilon > 0$ there is a gauge function $\delta: \Omega \rightarrow (0, \infty)$ such that for all δ -fine partitions $\{(z_i, \Omega_i)\}_{i=1}^N$ of Ω we have $|\sum_{i=1}^N f(z_i)\mu(\Omega_i) - \int_{\Omega} f d\mu| < \varepsilon$. We show in proving Theorem 12 how the properties of Lebesgue points can be used to determine the gauge δ . An even simpler result extending the Riemann integral is obtained in Section 5 for the case that f is continuous at μ -almost all points of Ω .

2 Covering Theorems Let $(X, \|\cdot\|)$ be a normed vector space of dimension $d < \infty$ over the real numbers \mathbb{R} . Then X is a separable, locally compact Hausdorff space with open sets determined by the open balls. The open ball with center $\mathbf{a} \in X$ and radius $r > 0$ is denoted by $U(\mathbf{a}, r) := \{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{a}\| < r\}$; the closed ball with center $\mathbf{a} \in X$ and radius $r > 0$ is $B(\mathbf{a}, r) := \{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{a}\| \leq r\}$. Since the dimension of X is finite, its closed balls are compact.

The integration results to follow will use coverings by balls and sets more general than balls in $(X, \|\cdot\|)$, and for this we will need the Besicovitch and Morse covering theorems. Strengthened versions of these theorems are as easy to state and prove as the original results. This was done at a real-analysis meeting in Rolla, Missouri in 1995, and the work was included in the report of that meeting as the note by the first author in [9]. Since all of that report is now essentially unavailable, we will sketch these improved results and proofs here for the reader's convenience.

For general finite dimensional normed vector spaces, the Besicovitch Covering Theorem uses covers by closed metric balls; it gives a constant that is independent of measure. Besicovitch's result is much stronger than the familiar Vitali Covering Theorem. It was originally established for disks in the plane in 1945-46 [1], and was extended by A. P. Morse [11] in 1947 to more general shapes in finite dimensional normed spaces. The constructions used in both the Besicovitch and Morse results are modified here so that better theorems are obtained. In the modified theorems, the original cover of a set A by closed sets can still be reduced to a subcover \mathcal{F} such that \mathcal{F} can be partitioned into n subfamilies of pairwise disjoint sets and n is bounded above by a global constant depending only on the space. The construction of \mathcal{F} is arranged, however, so that A is actually contained in the union of the interiors of the sets in \mathcal{F} . To obtain this result, we have modified the following definition taken from [2].

Definition 1 Fix $\tau > 1$. Let $\{S_i : 1 \leq i \leq n\}$ be an ordered collection of subsets of X with each S_i having finite diameter $\Delta(S_i)$ and containing a point a_i in its interior, $\text{int}(S_i)$. We say that the ordered collection of sets S_i is in τ -satellite configuration with respect to the

ordered set of points a_i if **i**) For all $i \leq n$, $S_i \cap S_n \neq \emptyset$ and **ii**) For all pairs $i < j \leq n$, $a_j \notin \text{int}(S_i)$ and $\Delta(S_j) < \tau \cdot \Delta(S_i)$.

Theorem 2 Let A be an arbitrary subset of X . With each point $a \in A$, associate a set $S(a)$ containing a in its interior so that the diameters have a finite upper bound. Assume that for some $\tau > 1$, there is an upper bound $\kappa \in \mathbb{N}$ to the cardinality of any ordered set $\{a_i : 1 \leq i \leq n\} \subseteq A$ with respect to which the ordered set $\{S(a_i) : 1 \leq i \leq n\}$ is in τ -satellite configuration. Then for some $m \leq \kappa$, there are pairwise disjoint subsets A_1, \dots, A_m of A such that $A \subseteq \bigcup_{j=1}^m \bigcup_{a \in A_j} \text{int}(S(a))$ and for each j , $1 \leq j \leq m$, the elements of the collection $\{S(a) : a \in A_j\}$ are pairwise disjoint.

Proof: Let T be a choice function on the nonempty subsets B of A such that $T(B)$ is a point $b \in B$ with $\tau \cdot \Delta(S(b)) > \sup_{a \in B} \Delta(S(a))$. Form a one-to-one correspondence between an initial segment of the ordinal numbers and a subcollection of A as follows. Set $B_1 = A$ and $a_1 = T(B_1)$. Having chosen a_α for $\alpha < \beta$, let $B_\beta = A \setminus \bigcup_{\alpha < \beta} \text{int}(S(a_\alpha))$. If $B_\beta \neq \emptyset$, set $a_\beta = T(B_\beta)$. There exists a first ordinal γ for which $B_\gamma = \emptyset$; that is, $A \subseteq \bigcup_{\alpha < \gamma} \text{int}(S(a_\alpha))$. Note that for $\alpha < \beta < \gamma$, we have $a_\beta \notin \text{int}(S(a_\alpha))$ and $\Delta(S(a_\beta)) < \tau \cdot \Delta(S(a_\alpha))$. Let $A_c = \{a_\alpha : \alpha < \gamma\}$, and let \prec denote the well-ordering on A_c inherited from the ordinals.

Given any nonempty subset B of A_c , form a one-to-one correspondence between an initial segment of the ordinal numbers and a subset $V(B)$ of B as follows. Set $B_1 = B$, and let $a(1)$ be the first element (with respect to \prec) of B_1 . Having chosen $a(\alpha)$ for $\alpha < \beta$, let

$$B_\beta = \{b \in B : \forall \alpha < \beta, S(b) \cap S(a(\alpha)) = \emptyset\}.$$

If $B_\beta \neq \emptyset$, let $a(\beta)$ equal the first element (with respect to \prec) of B_β . There exists a first ordinal γ for which $B_\gamma = \emptyset$. Let $V(B) = \{a(\alpha) : \alpha < \gamma\}$.

Now for $i \geq 1$, form sets $A_i \subseteq A_c$ as follows. Set $A_1 = V(A_c)$. Having chosen A_i for $1 \leq i \leq n$, let $B_n = A_c \setminus \bigcup_{i=1}^n A_i$. Stop if $B_n = \emptyset$. Otherwise, set $A_{n+1} = V(B_n)$. Note that for each $b \in B_n$ and each i between 1 and n , there is a first (with respect to \prec) $a_i \in A_i$ with $S(a_i) \cap S(b) \neq \emptyset$; clearly, $a_i \prec b$ in A_c . It now follows that the set $\{S(a_1), \dots, S(a_n), S(b)\}$ is in τ -satellite configuration with respect to the set $\{a_1, \dots, a_n, b\}$ when each set is given the ordering inherited from A_c . Therefore, $B_n = \emptyset$ for some $n \leq \kappa$. \square

Corollary 3 For any finite Borel measure μ on X , there is a j with $1 \leq j \leq m$ and a finite subset $A_\mu \subseteq A_j$ such that

$$\mu^*(A) \leq 2\kappa \cdot \sum_{a \in A_\mu} \mu(\text{int}(S(a))).$$

Proof: Take the first $j \leq m$ that maximizes the sum $\sum_{a \in A_j} \mu(\text{int}(S(a)))$. We can then choose a finite subset $A_\mu \subseteq A_j$ so that $\frac{1}{2} \cdot \sum_{a \in A_j} \mu(\text{int}(S(a))) \leq \sum_{a \in A_\mu} \mu(\text{int}(S(a)))$. \square

What is the upper bound κ for our vector space X ? For balls and values of τ close to 1, there is an upper bound K for κ established by Zoltán Füredi and the first author in [7]. It is the maximum number of points that can be packed into the closed ball $B(\mathbf{0}, 2)$ when one of the points is at $\mathbf{0}$ and the distance between distinct points is at least 1. That value is no more than 5^d , where d is the dimension of X . Applying Theorem 2 with $\kappa = K$ to a cover by balls yields an open version of Besicovitch's theorem for X . The constant K is the best constant for the Besicovitch Theorem in terms of all known proofs. With obvious modifications, the construction in [7] is already appropriate for the improved result that yields a cover by open balls.

To use Theorem 2 to establish an open version of Morse's Covering Theorem for $(X, \|\cdot\|)$, we need some geometric results. The proofs are modifications of arguments in [11] and [2]. The bound obtained is not as simple as the one for balls, but the shapes to which it applies are more general than balls or even convex sets. For these geometric arguments, we use boldface to denote points.

For each $\gamma \geq 1$, we let $N(\gamma)$ be an upper bound for the number of points that can be packed into the closed ball $B(\mathbf{0}, 1)$ when the distance between distinct points is at least $1/\gamma$ and one point is at $\mathbf{0}$. We write $N_S(\gamma)$ for the similar constant when all points are on the surface of $B(\mathbf{0}, 1)$. Given nonzero points \mathbf{b} and \mathbf{c} in X , we set $V(\mathbf{b}, \mathbf{c}) := \left\| \frac{\mathbf{b}}{\|\mathbf{b}\|} - \frac{\mathbf{c}}{\|\mathbf{c}\|} \right\|$.

Proposition 4 *Fix τ with $1 < \tau \leq 2$. Also fix an ordered set $\{S_i : 1 \leq i \leq n\}$ of bounded subsets of X each containing a ball $B(\mathbf{a}_i, r_i)$. Assume that $\{S_i : 1 \leq i \leq n\}$ is in τ -satellite configuration with respect to the ordered set of centers $\{\mathbf{a}_i : 1 \leq i \leq n\}$. Translate so that $\mathbf{a}_n = \mathbf{0}$. Fix $\lambda \geq \max_{1 \leq i \leq n} \Delta(S_i)/(2r_i)$. Suppose the resulting configuration has the following property in terms of two constants $C_0 \geq 1$ and $C_1 \geq 1$: If \mathbf{a}_i and \mathbf{a}_j are centers with the properties that $C_0 r_n < \|\mathbf{a}_i\| \leq \|\mathbf{a}_j\|$ and $V(\mathbf{a}_i, \mathbf{a}_j) \leq 1/C_1$, then \mathbf{a}_i must be in the interior of S_j . It then follows that*

$$n \leq N(2\lambda C_0) + N(8\lambda^2) N_S(C_1).$$

Proof: Set $r := r_n$ and $S := S_n$. For $1 \leq i < j \leq n$, we have

$$\|\mathbf{a}_i - \mathbf{a}_j\| \geq r_i \geq \Delta(S_i)/(2\lambda) \geq \Delta(S)/(4\lambda) \geq r/(2\lambda).$$

Scaling by $1/(C_0 r)$, one sees that there can be at most $N(2\lambda C_0)$ indices i for which $\|\mathbf{a}_i\| \leq C_0 r$. We only have to show, therefore, that there are at most $N(8\lambda^2) N_S(C_1)$ indices in the set $J := \{j < n : C_0 r < \|\mathbf{a}_j\|\}$. Suppose $i \neq j$ are members of J with $\mathbf{a}_i \in \text{int}(S_j)$. Then $i < j$ and

$$\mathbf{a}_j \in B(\mathbf{a}_i, \Delta(S_j)) \subseteq B(\mathbf{a}_i, 2\Delta(S_i)) \subseteq B(\mathbf{a}_i, 4\lambda r_i).$$

Moreover, $\|\mathbf{a}_j - \mathbf{a}_i\| \geq r_i \geq r_i/(2\lambda)$. If also $j < k$ in J , and $\mathbf{a}_i \in \text{int}(S_k)$, then $\mathbf{a}_k \in B(\mathbf{a}_i, 4\lambda r_i)$ and

$$\|\mathbf{a}_k - \mathbf{a}_j\| \geq r_j \geq \Delta(S_j)/(2\lambda) \geq \|\mathbf{a}_j - \mathbf{a}_i\|/(2\lambda) \geq r_i/(2\lambda).$$

Scaling by $1/(4\lambda r_i)$, it follows that for each $i \in J$, the cardinality $\text{Card}\{j \in J : \mathbf{a}_i \in \text{int}(S_j)\} \leq N(8\lambda^2)$. Now construct $J' \subseteq J$ by induction as follows. Set $J_1 = J$. At the k^{th} step for $k \geq 1$, if J_k is empty, stop. Otherwise, choose the first $i_k \in J_k$ so that for all $j \in J_k$, $\|\mathbf{a}_{i_k}\| \leq \|\mathbf{a}_j\|$. Put i_k in J' . Form the set J_{k+1} by discarding from J_k the index i_k and all other indices j such that $\mathbf{a}_{i_k} \in \text{int}(S_j)$. Now, if $i \neq j$ in J' , $V(\mathbf{a}_i, \mathbf{a}_j) > 1/C_1$. Therefore, $\text{Card}(J') \leq N_S(C_1)$, and so $\text{Card}(J) \leq N(8\lambda^2) N_S(C_1)$. \square

Given $\lambda \geq 1$ and $\mathbf{a} \in X$, we let $\mathcal{S}_\lambda(\mathbf{a})$ denote the collection of all sets $S \subseteq X$ for which there exists an $r > 0$ such that $B(\mathbf{a}, r) \subseteq S \subseteq B(\mathbf{a}, \lambda r)$ and S is **starlike** with respect to every $\mathbf{y} \in B(\mathbf{a}, r)$. This means that for each $\mathbf{y} \in B(\mathbf{a}, r)$ and each $\mathbf{x} \in S$, the line segment $\alpha \mathbf{y} + (1 - \alpha)\mathbf{x}$, $0 \leq \alpha \leq 1$, is contained in S . This is the general shape considered by Morse in [11]. To improve his result, as well as for work in a later section, we will need the following fact about such a set S ; the result, along with the next theorem, will finish our proof of the "open" Morse's Covering Theorem.

Proposition 5 *If $\|\mathbf{y} - \mathbf{a}\| < r$, i.e., if \mathbf{y} is in the interior of $B(\mathbf{a}, r)$, and \mathbf{x} is in the closure, $\text{cl}(S)$, of S , then every point of the form $\alpha\mathbf{y} + (1 - \alpha)\mathbf{x}$, $0 < \alpha \leq 1$, is in the interior of S .*

Proof: Fix $\rho > 0$ so that $B(\mathbf{y}, \rho) \subset B(\mathbf{a}, r)$, and fix α with $0 < \alpha \leq 1$. Assume first that $\mathbf{x} \in S$, and translate so that $\mathbf{x} = \mathbf{0}$. Then the ball $B(\alpha\mathbf{y}, \alpha\rho) \subseteq S$ since

$$\begin{aligned} \|\alpha\mathbf{y} - \mathbf{z}\| \leq \alpha\rho &\Rightarrow \left\| \mathbf{y} - \frac{1}{\alpha}\mathbf{z} \right\| \leq \rho \Rightarrow \frac{1}{\alpha}\mathbf{z} \in B(\mathbf{a}, r) \\ &\Rightarrow \mathbf{z} = \alpha \left(\frac{1}{\alpha}\mathbf{z} \right) + (1 - \alpha)\mathbf{0} \in S. \end{aligned}$$

Now for the case that $\mathbf{x} \in \text{cl}(S)$, choose a point $\mathbf{w} \in S$ so that $\frac{1-\alpha}{\alpha} \|\mathbf{x} - \mathbf{w}\| < \rho$. The result follows from the previous case since

$$\alpha\mathbf{y} + (1 - \alpha)\mathbf{x} = \alpha \left(\mathbf{y} + \frac{1-\alpha}{\alpha} (\mathbf{x} - \mathbf{w}) \right) + (1 - \alpha)\mathbf{w}. \quad \square$$

Theorem 6 *Fix $\lambda \geq 1$ and fix τ with $1 < \tau \leq 2$. If $\{S_i : 1 \leq i \leq n\}$ is an ordered collection of subsets of X in τ -satellite configuration with respect to an ordered set $\{\mathbf{a}_i : 1 \leq i \leq n\} \subset X$, and if for $1 \leq i \leq n$, $S_i \in \mathcal{S}_\lambda(\mathbf{a}_i)$, then*

$$n \leq N(64\lambda^3) + N(8\lambda^2)N_S(16\lambda).$$

Proof: For $1 \leq i \leq n$, fix $r_i > 0$ so that $B(\mathbf{a}_i, r_i) \subseteq S_i \subseteq B(\mathbf{a}_i, \lambda r_i)$ and S_i is starlike with respect to every $\mathbf{y} \in B(\mathbf{a}_i, r_i)$. Translate so that $\mathbf{a}_n = \mathbf{0}$; set $r = r_n$ and $S = S_n$. Suppose i and j are indices such that $32\lambda^2 r < \|\mathbf{a}_i\| \leq \|\mathbf{a}_j\|$ and $V(\mathbf{a}_i, \mathbf{a}_j) \leq 1/(16\lambda)$. By Proposition 4, we only have to show that \mathbf{a}_i must be in the interior of S_j . To simplify notation, let $\mathbf{b} = \mathbf{a}_i$ and $\mathbf{c} = \mathbf{a}_j$. Fix $\mathbf{x} \in S \cap S_j$. Since $\|\mathbf{x}\| \leq \lambda r < 32\lambda^2 r < \|\mathbf{b}\|$, $\mathbf{x} \neq \mathbf{b}$. Let $s = \|\mathbf{c}\|/\|\mathbf{b}\|$ and $t = 1/s$. Set $\mathbf{y} = (1 - s)\mathbf{x} + s\mathbf{b}$. Then $\mathbf{b} = (1 - t)\mathbf{x} + t\mathbf{y}$. To show that $\mathbf{b} \in \text{int}(S_j)$, we only have to show that $\|\mathbf{y} - \mathbf{c}\| < r_j$. Now $16\lambda\Delta(S) \leq 32\lambda^2 r < \|\mathbf{b}\|$, whence $\|\mathbf{x}\| \leq \Delta(S) \leq \min(\|\mathbf{b}\|/(16\lambda), 2\Delta(S_j))$. Therefore, since $|1 - s| = s - 1 < s$,

$$\begin{aligned} \|\mathbf{y} - \mathbf{c}\| &= \left\| (1 - s)\mathbf{x} + \|\mathbf{c}\| \left(\frac{\mathbf{b}}{\|\mathbf{b}\|} - \frac{\mathbf{c}}{\|\mathbf{c}\|} \right) \right\| \\ &< s\|\mathbf{x}\| + \|\mathbf{c}\|/(16\lambda) \\ &\leq s\|\mathbf{b}\|/(16\lambda) + \|\mathbf{c}\|/(16\lambda) = \|\mathbf{c}\|/(8\lambda) \\ &\leq (\|\mathbf{c} - \mathbf{x}\| + \|\mathbf{x}\|)/(8\lambda) \\ &< \Delta(S_j)/(2\lambda) \leq r_j. \quad \square \end{aligned}$$

3 Measures Recall that we are working with a normed vector space $(X, \|\cdot\|)$ of dimension $d < \infty$ over the real numbers \mathbb{R} . Let μ be a measure on a σ -algebra \mathcal{M} of subsets of X . We say that μ is a **Radon measure** on X if:

- (i) All Borel sets are measurable, i.e., \mathcal{M} contains the Borel sets.
- (ii) Compact sets have finite measure.
- (iii) μ is **inner and outer regular**, i.e., for all $E \in \mathcal{M}$

$$\begin{aligned} \mu(E) &= \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}, \\ \mu(E) &= \inf\{\mu(G) : G \supseteq E \text{ and } G \text{ is open}\}. \end{aligned}$$

We will call a set or function μ -**measurable**, or when μ is understood just **measurable**, if it is measurable with respect to the μ -completion of \mathcal{M} .

Since every open set in X is σ -compact, inner and outer regularity follow from assuming merely that μ is a Borel measure on X and closed balls have finite measure; see Theorem 2.18

in [12]. (For general spaces, the requirement of inner regularity is restricted to open sets and sets of finite measure; see Theorem 2.14 in [12].)

Given $\lambda \geq 1$ and $a \in X$, we say that a set $S_\lambda(a) \subseteq X$ is a **Morse set** associated with a and λ if there is an $r > 0$ such that $B(a, r) \subseteq S_\lambda(a) \subseteq B(a, \lambda r)$ and $S_\lambda(a)$ is starlike with respect to $B(a, r)$. We also say that $S_\lambda(a)$ is a λ -**Morse set**. Recall that a gauge function is a mapping $\delta : X \rightarrow (0, R)$ for some $R > 0$. We will say that the Morse set $S_\lambda(a)$ is δ -**fine** with respect to a gauge function δ if $\lambda r \leq \delta(a)$; in this case, we will also call a the **tag** for $S_\lambda(a)$. Note that putting $\lambda = 1$ forces a Morse set to be a closed ball. Also note that the closure $\text{cl}(S_\lambda(a))$ of a λ -Morse set $S_\lambda(a)$ is again a λ -Morse set since when $y \in B(a, r)$, $x \in \text{cl}(S_\lambda(a))$ and $\{x_n\}$ is a sequence converging to x , we have for any $\alpha \in [0, 1]$, $\alpha y + (1 - \alpha)x_n \rightarrow \alpha y + (1 - \alpha)x$.

A collection $\mathcal{S} \subseteq \mathcal{P}(X)$ consisting of at least one Morse set associated with each point a in a set $\Omega \subseteq X$ is called a **Morse cover** of Ω provided the same $\lambda \geq 1$ is used for each set in the cover and there is a finite upper bound to the diameters of the sets in the cover. We will also call such a cover a λ -**Morse cover**. A λ -Morse cover \mathcal{S} of Ω is called **fine** if for each $a \in \Omega$ and arbitrarily small values of $r > 0$ there are associated sets $S_\lambda(a) \in \mathcal{S}$ with $B(a, r) \subseteq S_\lambda(a) \subseteq B(a, \lambda r)$ such that $S_\lambda(a)$ is starlike with respect to $B(a, r)$. Given a Radon measure μ , a λ -Morse cover of a measurable set $\Omega \subseteq X$ is called a μ -**a.e. cover** of Ω if **i)** it is fine, **ii)** each set in the cover is μ -measurable, and **iii)** for any $\varepsilon > 0$, and any strictly positive gauge function δ there is a finite or infinite sequence of disjoint, δ -fine sets $S_n \in \mathcal{S}$ such that $\mu(\Omega \setminus \cup_n S_n) = 0$ and $\mu(\cup_n S_n \setminus \Omega) < \varepsilon$. This concept is similar to that of Vitali covers, see [5].

We first extend Corollary 3 to show that a fine Morse cover consisting of closed sets is a μ -a.e. cover for any given measurable subset Ω of X . The same is true when the Morse sets are not necessarily closed provided that for each set E in the cover, it does not increase the measure of E to adjoin its closure points. For closed balls and sets of finite measure, the proof is standard (see [4] or [13]). We reproduce and extend it here.

Lemma 7 *Let μ be a Radon measure on X . Let $\Omega \subseteq X$ be measurable, and suppose that \mathcal{S} is a fine Morse cover of Ω consisting of μ -measurable sets. Then \mathcal{S} is a μ -a.e. cover of Ω if \mathcal{S} consists of closed sets or if for each set $E \in \mathcal{S}$, $\mu(\Omega \cap (\text{cl}(E) \setminus E)) = 0$.*

Proof: Fix $\varepsilon > 0$, and a gauge function $\delta > 0$. We suppose first that \mathcal{S} consists of closed, δ -fine sets. If $\mu(\Omega) < \infty$, we may fix an open set $O \supseteq \Omega$ such that $\mu(O \setminus \Omega) < \varepsilon$, and we may assume that each set $E \in \mathcal{S}$ is a subset of O . Let κ be the upper bound for the Morse Covering Theorem; recall that it depends only on X and the parameter λ for the cover. By Corollary 3, there is a finite subcollection $\mathcal{F}_1 \subset \mathcal{S}$ consisting of pairwise disjoint closed sets such that $\mu(\cup \mathcal{F}_1) \geq \mu(\Omega)/(2\kappa)$, whence $\mu(\Omega \setminus \cup \mathcal{F}_1) \leq (1 - 1/(2\kappa))\mu(\Omega)$. Let $\Omega' = \Omega \setminus \cup \mathcal{F}_1$ and $\mathcal{S}_1 = \{E \in \mathcal{S} : E \cap (\cup \mathcal{F}_1) = \emptyset\}$. Then \mathcal{S}_1 is a fine Morse cover of Ω' . Again, there is a finite disjoint subfamily $\mathcal{F}_2 \subseteq \mathcal{S}_1$ such that $\mu(\Omega' \setminus \cup \mathcal{F}_2) \leq (1 - 1/(2\kappa))\mu(\Omega')$, whence, $\mu(\Omega \setminus (\cup(\mathcal{F}_1 \cup \mathcal{F}_2))) \leq (1 - 1/(2\kappa))^2 \mu(\Omega)$. Continuing in this manner, we have $\mu(\Omega \setminus \mathcal{F}) = 0$ where $\mathcal{F} = \cup_i \mathcal{F}_i$. Important for the next step, however, is the fact that for any $\gamma > 0$, there is a finite, pairwise disjoint family $\mathcal{F}' \subseteq \mathcal{S}$ such that $\mu(\Omega \setminus \mathcal{F}') < \gamma$.

Now suppose that $\mu(\Omega) = +\infty$. Then since μ is a Radon measure, $\Omega = \cup_{i=1}^{\infty} \Omega_i$ where each Ω_i is a set of finite measure and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. For each i , fix an open set $O_i \supseteq \Omega_i$ with $\mu(O_i \setminus \Omega_i) < \varepsilon/2^i$. We apply the above result to obtain a finite (or empty) family \mathcal{F}^1 covering all but a set of measure 1 of Ω_1 with all sets contained in O_1 . At the n^{th} stage, $n > 1$, we obtain a finite (or empty) family \mathcal{F}^n covering all but a set of measure $1/n$ of $(\cup_{i=1}^n \Omega_i) \setminus \cup_{i=1}^{n-1} (\cup \mathcal{F}^i)$ with all sets contained in $(\cup_{i=1}^n O_i) \setminus \cup_{i=1}^{n-1} (\cup \mathcal{F}^i)$. Clearly, $\cup_{i=1}^{\infty} \mathcal{F}^i$ is the desired collection of disjoint sets in \mathcal{S} .

In the case that for each set $E \in \mathcal{S}$, $\mu(\Omega \cap (\text{cl}(E) \setminus E)) = 0$, we apply the above result to the Morse cover formed by the closures of the sets in \mathcal{S} . We then replace each set $\text{cl}(S_n)$ in the resulting disjoint sequence with the original set S_n . \square

When dealing with Morse sets that are not closed, the conditions in Lemma 7 are easily fulfilled when the Morse cover \mathcal{S} is **scaled**. This means that for each $S_\lambda(a) \in \mathcal{S}$ and each $p \in (0, 1]$, the set $S_\lambda^{(p)}(a)$ is also in \mathcal{S} where $S_\lambda^{(p)}(a) = \{a + px : a + x \in S_\lambda(a)\}$.

Proposition 8 *Let μ be a Radon measure on X . Let Ω be a measurable subset of X and suppose \mathcal{S} is a scaled Morse cover of Ω consisting of μ -measurable sets. Then \mathcal{S} is a μ -a.e. cover of Ω .*

Proof: Since \mathcal{S} is a scaled Morse cover of Ω , it is certainly a fine cover of Ω . Let λ be the parameter for the Morse cover \mathcal{S} . Let $a \in \Omega$ and fix $S_\lambda(a) \in \mathcal{S}$; we write S for $S_\lambda(a)$. We will show that for $0 < p < q \leq 1$, $\partial S^{(p)} \cap \partial S^{(q)} = \emptyset$. The result will then follow since for all but a countable number of values p , $\mu(\partial S^{(p)}) = 0$. Since $S^{(p)} = (S^{(q)})^{(p/q)}$, we may simplify notation by assuming that $S^{(q)} = S$; we may further simplify by translating so that $a = 0$. The result now follows from Proposition 5 since for each $x \in \text{cl}(S^{(p)})$, $(1/p)x \in \text{cl}(S)$, so $x \in \text{int}(S)$. \square

Example 9 Take all closed balls or all open balls in X of radius at most 1. For each center x and radius r , let $a(x, r)$ in the interior of the ball be the tag of that ball, and set $\omega(x, r) := \|x - a(x, r)\|/r$. Assume that $\omega_0 = \sup_{x, r} \omega(x, r) < 1$. Given a Radon measure μ , we have a μ -a.e. cover of any μ -measurable set in X , and $(1 + \omega_0)/(1 - \omega_0)$ is the smallest permissible value of λ . As a special case, we may take each tag $a(x, r) = x$.

Example 10 Let $\{e_1, \dots, e_d\}$ be a basis for X . Let $a = \sum_{i=1}^d a_i e_i \in X$. Let $b, c \in \mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ with $c_1^2 + \dots + c_d^2 < 1$. Define a tagged interval by setting $I(a, b, c) := \{\sum_{i=1}^d (a_i + t_i) e_i : 0 < t_i \leq b_i, 1 \leq i \leq d\}$ with tag at $\sum_{i=1}^d (a_i + b_i c_i) e_i$. Fix c as above and take $k \geq 1$. Given a Radon measure μ , the collection $\mathcal{S} = \{I(a, b, c) : a \in X, b \in \mathbb{R}_+^d \text{ such that } \max_{1 \leq i \leq d} b_i / \min_{1 \leq i \leq d} b_i \leq k\}$ is a scaled, μ -a.e. Morse cover of X .

Let K be a compact subset of X and let μ be a Radon measure such that each open ball with center at a point of K has positive μ -measure. We will want to use the fact that given a $\lambda \geq 1$, any μ -a.e., λ -Morse cover \mathcal{S} of K forms a differentiation basis on K with respect to μ . For our purposes here this means that if ν is a radon measure absolutely continuous with respect to μ , i.e., $\nu \ll \mu$, and $\mathcal{S}(a)$ is the collection of sets in \mathcal{S} associated with $a \in K$, then

$$\lim_{\substack{S \in \mathcal{S}(a) \\ \Delta(S) \rightarrow 0}} \frac{\nu(S)}{\mu(S)} = \frac{d\nu}{d\mu}(a) \text{ for } \mu\text{-a.e. } a \in K,$$

where $\frac{d\nu}{d\mu}$ denotes the Radon-Nikodým derivative of ν with respect to μ .

By the principal result in [2], the above equality follows from the fact that if E is a measurable subset of K and ν is a finite Radon measure with $\nu \ll \mu$ and $\nu(E) = 0$, then for μ -a.e. $a \in E$, $\limsup_{S \in \mathcal{S}(a), \Delta(S) \rightarrow 0} \nu(S)/\mu(S) \leq 1$. As in [2], we can see that this is in fact the case by letting A be the subset of E where the reverse inequality holds, and letting κ be the upper bound given by the Morse Covering Theorem. We fix $\varepsilon > 0$ and a nonempty compact set $C \subset X \setminus E$ with $\nu(X \setminus C) < \varepsilon/(2\kappa)$. By assumption, for each $a \in A$, there

is a set $S(a) \in \mathcal{S}(a)$ with $S(a) \cap C = \emptyset$ and $\mu(S(a)) \leq \nu(S(a))$. For the finite, disjoint subcollection $\langle S_n \rangle$ of these sets given by Corollary 3, we have

$$\mu^*(A) \leq 2\kappa \cdot \sum_n \mu(S_n) \leq 2\kappa \cdot \sum_n \nu(S_n) \leq 2\kappa \cdot \nu(X \setminus C) < \varepsilon.$$

In the next section, we will want to exploit the fact that measurable functions are approximately continuous almost everywhere with respect to a given Radon measure μ . That is, let Ω be a μ -measurable subset of X , and let $f: \Omega \rightarrow \mathbb{R}$ be μ -measurable; set $f \equiv 0$ on $X \setminus \Omega$. Suppose \mathcal{S} is a fine λ -Morse cover of Ω , so that the sets in \mathcal{S} form a differentiation basis with respect to μ at points $x \in \Omega$ for which all balls $B(x, r)$ have positive μ -measure. Then $x \in \Omega$ is called a **point of approximate continuity** of f if for all positive ε and η there is an $R > 0$ such that if $S(x)$ is a set in \mathcal{S} with tag x and $S(x) \subseteq B(x, R)$, then for $E(x, \eta) := \{y \in S(x) : |f(x) - f(y)| > \eta\}$ we have $\mu(E(x, \eta)) \leq \varepsilon \mu(S(x))$. It is known that μ -almost all points of Ω are points of approximate continuity of f (see [5], 2.9.13). A related notion, defined and used below in the proof of Theorem 12, is the notion of a **Lebesgue point** for f ; these also fill the space except for a set of measure 0.

Remark 11 Clearly, a nonnegative, measurable, real-valued function f is approximately continuous μ -a.e. if for each $n \in \mathbb{N}$, $\min(f, n+1)$ is approximately continuous μ -a.e. on the set where $f \leq n$. That this is the case follows from the discussion of Lebesgue points in Section 3 of [3], since the constant for a Lebesgue point x equals $f(x)$ for μ -almost all x (cf. Equation (2) below).

4 Integration Again, we let $(X, \|\cdot\|)$ be a normed vector space of dimension $d < \infty$ over the real numbers \mathbb{R} . Using our covering results we can formulate the Lebesgue integral as a type of Riemann sum defined by μ -a.e. Morse covers. We do this first for nonnegative functions and later apply the result to measurable functions taking both positive and negative values.

Theorem 12 *Let μ be a Radon measure on X . Let Ω be a measurable subset of X , and let f be a nonnegative, real-valued, measurable function on Ω . Then $\int_\Omega f d\mu$ is finite and equals F if the following condition holds for some $\lambda \geq 1$ and some μ -a.e., λ -Morse cover \mathcal{S} of Ω : For all $\varepsilon > 0$ there is a gauge function $\delta: \Omega \rightarrow (0, 1]$ such that for any finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω we have*

$$(1) \quad \left| \sum_n f(x_n) \mu(S_n) - F \right| < \varepsilon.$$

Conversely, if $\int_\Omega f d\mu$ is finite and equals F , then the condition holds for any $\lambda \geq 1$ and any μ -a.e., λ -Morse cover \mathcal{S} of Ω .

Proof: We note first that for a given set $A \subseteq \Omega$ with $\mu(A) = 0$, we may set our gauge to force an arbitrarily small sum for points $x_i \in A$, and also force, in the case that f is assumed to be integrable, an arbitrarily small integral of f over the union of the corresponding sets S_i . To show this, we fix $\varepsilon > 0$, and for each $n \in \mathbb{N}$ we set $A_n = \{x \in A : n-1 \leq f(x) < n\}$. The sets A_n are disjoint and μ -null with union A . In the case that f is assumed to be integrable, we may choose an open set $G \supseteq A$ so that $\int_G f < \varepsilon$; otherwise, set $G = X$. For each $n \in \mathbb{N}$, fix an open set G_n with $G \supseteq G_n \supseteq A_n$ and $\mu(G_n) < \varepsilon / (n \cdot 2^n)$. (This is possible since μ is outer regular.) For each $x \in A_n$, we choose $\delta(x) < \sup\{s : B(x, s) \subseteq G_n\}$. Then a sum over δ -fine, disjoint sets S_i with all tags in A satisfies the inequality

$$\sum_i f(x_i) \mu(S_i) < \sum_{n=1}^{\infty} \left(n \sum_{x_i \in A_n} \mu(S_i) \right) \leq \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon,$$

and if f is assumed to be integrable, its integral over $\cup_i S_i$ is at most ε .

Now suppose that $\int_{\Omega} f d\mu$ exists and equals F . Fix $\lambda \geq 1$, a μ -a.e., λ -Morse cover \mathcal{S} of Ω , and an $\varepsilon > 0$. Set $f \equiv 0$ on $X \setminus \Omega$. Suppose $x \in \Omega$ is a Lebesgue point for f with respect to the Morse cover \mathcal{S} . This means that there is a constant, which (after redefining f on a μ -null set) we may assume is $f(x)$, such that the following condition holds: For any $\varepsilon_1 > 0$ there is an $R > 0$ so that if $S(x)$ is a set in \mathcal{S} with tag x and $S(x) \subseteq B(x, R)$, then

$$(2) \quad \int_{y \in S(x)} |f(x) - f(y)| d\mu(y) \leq \varepsilon_1 \mu(S(x)).$$

For such an x , let $k(x)$ be the first integer strictly larger than $\|x\|$. Set $\delta(x) = R$ where R is chosen to be at most 1 and satisfy Equation (2) with $\varepsilon_1 = \varepsilon 2^{-k(x)-1} / [1 + \mu(B(0, k(x) + 1))]$. Since \mathcal{S} forms a differentiation basis, it follows that the non-Lebesgue points form a μ -null set. (See, for example, Section 3 of [3].) We may, as just noted, choose positive values $\delta(x) \leq 1$ for such points x so that their contribution to the sum in Equation (1) can be at most $\varepsilon/4$ and the integral of f over the union of the corresponding sets $S(x)$ will be at most $\varepsilon/4$.

With this choice for the gauge δ , we now let $\langle S_n(x_n) \rangle$ be any finite or countably infinite disjoint sequence of δ -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω . Let L denote the set of Lebesgue points of Ω . Then

$$(3) \quad \left| \int_{\Omega} f d\mu - \sum_n f(x_n) \mu(S_n) \right| = \left| \int_{\cup_n S_n} f d\mu - \sum_n f(x_n) \mu(S_n) \right| \\ \leq \sum_{x_n \in L} \int_{S_n} |f(x_n) - f(y)| d\mu(y) + \frac{\varepsilon}{2} \\ \leq \sum_{\ell=1}^{\infty} \frac{\varepsilon 2^{-\ell-1}}{1 + \mu(B(0, \ell + 1))} \sum_{\ell-1 \leq \|x_n\| < \ell} \mu(S_n) + \frac{\varepsilon}{2} \\ \leq \varepsilon.$$

Now fix a $\lambda \geq 1$ and a μ -a.e., λ -Morse cover \mathcal{S} of Ω so that for any $\varepsilon > 0$ there is an appropriate gauge $\delta \leq 1$ for f and F ; that is, for any finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω , Equation (1) holds for ε . For each $x \in \Omega$, let $k(x)$ be the first integer strictly larger than $\|x\|$, and set

$$\eta(x) := \frac{2^{-k(x)}}{[1 + \mu(B(0, k(x) + 1))][1 + f(x)]}.$$

For each $m \in \mathbb{N}$, fix $\delta_m \leq 1$ to work for f and F with $\varepsilon = 1/m$ in Equation (1). Let $\langle S_n^m(x_n^m) \rangle$ be a finite or countably infinite disjoint sequence of δ_m -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω . We may assume that each tag x_n^m is a point of approximate continuity of f and $\mu(E_n^m) \leq \eta_m(x_n^m) \mu(S_n^m)$ where $\eta_m(x_n^m) = \eta(x_n^m)/m$ and

$$E_n^m := \{x \in S_n^m : |f(x_n^m) - f(x)| > \eta_m(x_n^m)\}.$$

Define a measurable function f_m on Ω as follows: If for some $n \in \mathbb{N}$, $x \in S_n^m \setminus E_n^m$, set $f_m(x) = \max(f(x_n^m) - \eta_m(x_n^m), 0)$; otherwise, set $f_m(x) = 0$. Now the functions f_m converge

to f in measure since,

$$\begin{aligned}
& \mu \left(\left\{ x \in \Omega : |f(x) - f_m(x)| > \frac{1}{m} \right\} \right) \\
& \leq \sum_n \mu(E_n^m) \leq \sum_n \eta_m(x_n^m) \mu(S_n^m) \\
& \leq \frac{1}{m} \sum_{\ell=1}^{\infty} \frac{2^{-\ell}}{1 + \mu(B(0, \ell + 1))} \sum_{\ell-1 \leq \|x_n^m\| < \ell} \mu(S_n^m) \\
& \leq \frac{1}{m}.
\end{aligned}$$

Since any subsequence of the sequence $\langle f_m \rangle$ has in turn a subsequence converging μ -a.e. to f , it follows from Fatou's lemma that

$$\begin{aligned}
\int_{\Omega} f d\mu & \leq \liminf_m \int_{\Omega} f_m d\mu \\
& \leq \liminf_m \sum_n f(x_n^m) \mu(S_n^m) \\
& \leq \liminf_m (F + 1/m) \\
& = F < +\infty.
\end{aligned}$$

On the other hand, each $f_m \leq f$, so for each m ,

$$\begin{aligned}
\int_{\Omega} f d\mu & \geq \int_{\Omega} f_m d\mu \\
& \geq \sum_n [f(x_n^m) - \eta_m(x_n^m)] \mu(S_n^m \setminus E_n^m) \\
& = \sum_n f(x_n^m) \mu(S_n^m) - \sum_n f(x_n^m) \mu(E_n^m) - \sum_n \eta_m(x_n^m) \mu(S_n^m \setminus E_n^m) \\
& \geq F - 1/m - \sum_n \eta_m(x_n^m) f(x_n^m) \mu(S_n^m) - \sum_n \eta_m(x_n^m) \mu(S_n^m) \\
& \geq F - 1/m - \frac{1}{m} \sum_{\ell} \frac{2^{-\ell}}{1 + \mu(B(0, \ell + 1))} \sum_{\ell-1 \leq \|x_n^m\| < \ell} 2 \mu(S_n^m) \\
& \geq F - 3/m,
\end{aligned}$$

whence $\int_{\Omega} f d\mu = F$. \square

Remark 13 With no loss of generality, we can restrict the points x_n in Equation (1) to be points of approximate continuity or to be points outside of any given μ -null set. Also, while we could work with the cover formed by all μ -measurable λ -Morse sets, the gauge δ can in general be chosen larger when given a smaller μ -a.e. Morse cover.

Let f be a real-valued function on Ω taking both positive and negative values. As usual, we set $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$; given μ , we say that f is integrable if both f^+ and f^- have finite integrals with respect to μ . Suppose now that \mathcal{S} is the set of all closed balls in X with tags at the center; i.e., $\lambda = 1$. Even for this case, we cannot force the integrability of f with the inequality

$$(4) \quad \left| \sum_{n=1}^{\infty} f(x_n) \mu(B_n) - F \right| < \varepsilon.$$

The inequality does imply that $\sum |f(x_n)|\mu(B_n)$ will be bounded for any appropriate sequence $\langle B_n \rangle$, but the sums need not be uniformly bounded. The condition given by (4) will allow principal value integrals. For example, in \mathbb{R}^d let e_1 be the unit vector in the positive direction along the first axis. For each $n \in \mathbb{N}$, let A_n be the open ball $U(((-1)^n/n) \cdot e_1, 1/(2n^2))$. The balls A_n are disjoint. Let Ω be the union of the balls A_n together with the origin, and let μ be the sum of the Dirac measure supported at the origin and Lebesgue measure restricted to Ω . Set $f(x) = ((-1)^n/n) \cdot \mu(A_n)$ if $x \in A_n$, and let $f(0) = 0$. Take the gauge function $\delta: \mathbb{R}^d \rightarrow (0, 1)$ so that if $x \in A_n$ then $B(x, \delta(x)) \subset A_n$. Let $F = \sum_{n=1}^{\infty} (-1)^n/n$, i.e., $F = -\ln 2$. Given $\varepsilon > 0$, if we take δ small enough at 0, then for any sequence of disjoint, δ -fine balls B_n satisfying $\mu(\Omega / \cup_n B_n) = 0$, we have $|\sum f(x_n)\mu(B_n) - F| < \varepsilon$. Any such sequence must contain a ball having the origin as its center. As we choose different sequences so that the radius of this ball shrinks to 0 we have $\sum |f(x_n)|\mu(B_n) \rightarrow \infty$.

It is the case, as we now show, that a real-valued, measurable f is integrable if the sums $\sum_n |f(x_n)| \mu(S_n)$ are uniformly bounded.

Corollary 14 *Given μ and Ω as in the theorem, let f be an arbitrary, measurable, real-valued function on Ω . Then f is integrable if the following condition holds for some $\lambda \geq 1$ and some μ -a.e., λ -Morse cover \mathcal{S} of Ω : There is a number $M \geq 0$ and a gauge function $\delta: \Omega \rightarrow (0, 1]$ such that for any finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω we have*

$$(5) \quad \sum_n |f(x_n)| \mu(S_n) \leq M.$$

Conversely, if f is integrable, then the condition holds for all $\lambda \geq 1$ and all μ -a.e., λ -Morse covers \mathcal{S} of Ω . In this case, for each such λ -Morse cover \mathcal{S} and each $\varepsilon > 0$, there is a gauge function $\delta: \Omega \rightarrow (0, 1]$ so that for any finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω we have

$$\left| \sum_n f(x_n) \mu(S_n) - \int_{\Omega} f d\mu \right| < \varepsilon.$$

Proof: Fix a $\lambda \geq 1$ and a μ -a.e., λ -Morse cover \mathcal{S} of Ω , and suppose there is an $M \geq 0$ and a gauge $\delta \leq 1$ satisfying our condition including Equation (5). For each $x \in \Omega$, let $k(x)$ be the first integer strictly larger than $\|x\|$, and set $\eta(x) := 2^{-k(x)} / (1 + \mu(B(0, k(x) + 1)))$. For each $m \in \mathbb{N}$, let $\langle S_n^m(x_n^m) \rangle$ be a finite or countably infinite disjoint sequence of δ -fine sets from \mathcal{S} covering all but a set of measure 0 of Ω . We may assume that each tag x_n^m is a point of approximate continuity of $|f|$ and $\mu(E_n^m) \leq \eta_m(x_n^m) \mu(S_n^m)$ where $\eta_m(x_n^m) = \eta(x_n^m)/m$ and

$$E_n^m := \{x \in S_n^m : \left| |f(x_n^m)| - |f(x)| \right| > \eta_m(x_n^m)\}.$$

Define a measurable function f_m on Ω as follows: If for some $n \in \mathbb{N}$, $x \in S_n^m \setminus E_n^m$, set $f_m(x) = \max(|f(x_n^m)| - \eta_m(x_n^m), 0)$; otherwise, set $f_m(x) = 0$. As in the theorem, we have $f_m \rightarrow |f|$ in measure and

$$\int_{\Omega} |f| d\mu \leq \liminf \int_{\Omega} f_m d\mu \leq \liminf \sum_n |f(x_n^m)| \mu(S_n^m) \leq M,$$

whence f is integrable.

Now assume that f is integrable, and set $F_1 = \int_{\Omega} |f| d\mu$. Applying the theorem, it follows that for any $\lambda \geq 1$ and any μ -a.e., λ -Morse cover \mathcal{S} of Ω , the function $|f|$ satisfies our condition including Equation (5) with $M = F_1 + 1$. The rest follows for any given $\varepsilon > 0$ by applying the theorem separately to f^+ and f^- with respect to $\varepsilon/2$ and taking the smaller of the two gauges at each point. \square

5 An Extension of the Riemann Integral For the case that f is real-valued and continuous almost everywhere, we can easily calculate the gauge δ , and in the process obtain an extension of the Riemann integral that integrates some unbounded functions with respect to Radon measures on unbounded domains. Here too, we say that f is integrable only when this is true for f^+ and f^- .

Theorem 15 *Let μ be a Radon measure on X . Let Ω be a measurable subset of X , and let f be a measurable, real-valued function on Ω . Set $f \equiv 0$ on $X \setminus \Omega$, and let Ω_c be the set of points in Ω where f is continuous. Let us suppose that $\mu(\Omega \setminus \Omega_c) = 0$. For $x \in \Omega_c$, let $k(x)$ be the smallest integer strictly greater than $\|x\|$, and for each $\gamma > 0$ fix $\rho(x, \gamma)$ with $0 < \rho(x, \gamma) \leq 1$ so that for all y with $\|y - x\| < \rho(x, \gamma)$, we have $|f(y) - f(x)| < \gamma$. If $\mu(\Omega) < \infty$, then for each $\varepsilon > 0$ and each $x \in \Omega_c$ set $\delta_\varepsilon(x) = \rho(x, \varepsilon \cdot [1 + \mu(\Omega)]^{-1})$; otherwise for each $\varepsilon > 0$ and each $x \in \Omega_c$ set $\delta_\varepsilon(x) = \rho(x, \varepsilon \cdot 2^{-k(x)} \cdot [1 + \mu(B(0, k(x) + 1))])^{-1}$. Now, if f is integrable, then for any $\lambda \geq 1$, any ε with $0 < \varepsilon \leq 1$ and any finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ_ε -fine, λ -Morse sets covering all but a set of measure 0 of Ω and having tag points $x_n \in \Omega_c$ we have*

$$\left| \sum_n f^\pm(x_n) \mu(S_n(x_n)) - \int_\Omega f^\pm d\mu \right| < \varepsilon,$$

whence

$$\left| \sum_n f(x_n) \mu(S_n(x_n)) - \int_\Omega f d\mu \right| < 2\varepsilon.$$

On the other hand, f is integrable if for some $\lambda \geq 1$ and some finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ_1 -fine Morse sets, associated with λ and tag points $x_n \in \Omega_c$, and covering all but a set of measure 0 of Ω , we have

$$\sum_n |f(x_n)| \mu(S_n) < +\infty.$$

Proof: Note that if $\rho(x, \gamma)$ works for f , then it works for f^+ and f^- . Assume f is integrable, and fix $\lambda \geq 1$ and $\varepsilon > 0$. Let $\langle S_n(x_n) \rangle$ be any finite or countably infinite disjoint sequence of δ_ε -fine, λ -Morse sets with tag points x_n in Ω_c and covering all but a set of measure 0 of Ω . Then for the case that $\mu(\Omega) = \infty$ we have

$$\begin{aligned} \left| \int_\Omega f^+ d\mu - \sum_n f^+(x_n) \mu(S_n) \right| &\leq \sum_n \int_{S_n} |f^+(y) - f^+(x_n)| d\mu(y) \\ &\leq \sum_{\ell=1}^{\infty} \frac{\varepsilon 2^{-\ell}}{1 + \mu(B(0, \ell + 1))} \sum_{\ell-1 \leq \|x_n\| < \ell} \mu(S_n) \leq \varepsilon, \end{aligned}$$

with the obvious simplification for the case that $\mu(\Omega) < \infty$. A similar calculation works for f^- .

Now fix $\lambda \geq 1$, and assume there is a finite or countably infinite disjoint sequence $\langle S_n(x_n) \rangle$ of δ_1 -fine Morse sets associated with λ and tag points $x_n \in \Omega_c$ covering all but a set of measure 0 of Ω such that $\sum_n |f(x_n)| \mu(S_n) = M \in \mathbb{R}$. Then for the case that

$$\mu(\Omega) = \infty,$$

$$\begin{aligned} \int_{\Omega} |f| \, d\mu &= \sum_n \int_{S_n} |f| \, d\mu \\ &\leq \sum_{\ell} \left(|f(x_n)| + \frac{2^{-\ell}}{1 + \mu(B(0, \ell + 1))} \right) \sum_{\ell-1 \leq \|x_n\| < \ell} \mu(S_n) \\ &\leq M + 1. \end{aligned}$$

Again, we have the obvious simplification for the case that $\mu(\Omega) < \infty$. \square

Note added in proof: It follows from Lusin's Theorem and the Lebesgue Differentiation Theorem for characteristic functions that this theory can be extended to Banach space valued functions. This will be the subject of a subsequent paper.

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