BEST FILTERS FOR THE GENERAL FATOU BOUNDARY LIMIT THEOREM

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ABSTRACT. Given a suitable normalization, there is a "best" family of filters for which the Fatou Boundary Limit Theorem holds. The normalization assigns to each positive harmonic function a set of boundary points at which that function must vanish. Known limits, such as those provided by the Lebesgue Differentiation Theorem, are used to force consistency in this assignment. The zero sets, in turn, are used in constructing the coarsest filters which produce those limits almost everywhere. This procedure is formulated in terms of a general potential theoretic setting and a general reference measure. The result is new, however, even for harmonic functions on the unit disk.

1. INTRODUCTION

In [4], J. L. Doob showed that for positive harmonic functions on the unit disk, there is no "best", i.e., coarsest, filter for which a Fatou Boundary Limit Theorem holds when the filter is copied by rotation at all points of the unit circle. In this note we use the principal result from [3] to show that after a suitable normalization of the problem, there is a "best" family of filters for approaching the boundary. The normalization assigns to each positive harmonic function a set of boundary points at which that function must vanish. Known limits, such as those given by the Lebesgue Differentiation Theorem, guide in making this assignment. The zero sets are then used in constructing a filter of approach neighborhoods at each boundary point. These filters are the coarsest ones for which a Fatou Boundary Limit Theorem holds and the required zero limits are achieved.

The result we use from [3] is a necessary and sufficient condition for almost everywhere convergence; the corresponding filters are the smallest collections of large neighborhoods producing the desired limits. This suggests that sets from other filters, such as nontangential neighborhoods, work because they are subsets of the neighborhoods constructed here.

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Indeed, in retrospect, one finds this containment at the heart of the short proof of the fine limit theorem given by the authors in [2].

Our principal result will be stated for very general, potential theoretic settings and general reference measures. The result is new, however, even for harmonic functions on the unit disk. One application for the disk, or any domain with the Martin boundary, is a construction of filters strictly coarser than those formed from minimal fine neighborhoods.

2. Fatou-filter mappings

Let X be a compactifying boundary for a connected and locally connected, locally compact, but not compact Hausdorff space W. We assume that W is the domain of definition for "harmonic functions" in a linear potential theoretic setting. Appropriate examples are a domain in a Euclidean space or a Riemann surface with ordinary harmonic functions, and even a domain in an axiomatic, potential theoretic setting (see [5], [1], and [7]). Let $\mathcal{B}$ denote the Borel subsets of X, and fix a finite reference measure σ on $(X, \mathcal{B})$. Given any finite Borel measure ν and a set $E \in \mathcal{B}$, we write ν_E for the "restriction" of ν to E; i.e., $\nu_E(A) = \nu(A \cap E)$ for each $A \in \mathcal{B}$. We assume that X supports an integral representation in the following sense:

(1) There is a convex cone M of finite Borel measures on X, with $\sigma \in M$ and $\nu_E \in M$ when $\nu \in M$ and $E \in \mathcal{B}$.
(2) To each $\nu \in M$, there corresponds a harmonic function h_ν on W so that the map $\nu \mapsto h_\nu$ is a positive affine map on M.
(3) The function h_σ is strictly positive on W.
(4) If $\nu \in M$ and $h_\sigma \le h_\nu$, then $\sigma \le \nu$; i.e., for each $E \in \mathcal{B}$, $\sigma(E) \le \nu(E)$.

2.1 Example. A familiar example of such an integral representation is obtained by letting X and W be the unit circle C and unit disk D, respectively, in the complex plane, letting σ be normalized Lebesgue measure on C, setting M equal to the set of finite Borel measures on C, and letting h_ν denote the Poisson integral of ν for each $\nu \in M$.

2.2 Example. Other integral representations can be found by letting X be the Martin boundary [5], or the boundary constructed by the second author in [7], for the space of ordinary harmonic functions on a domain in Euclidean space or for a more general space of harmonic functions (see [5], [1], and [7]). Of course, one can always restrict M to the set of all measures absolutely continuous with respect to the reference measure σ, or to measures generated by densities in L^p for some p with $1 \le p \le \infty$.

Given $\nu \in M$, we write $d\nu/d\sigma$ for the Radon-Nikodym derivative of the absolutely continuous part of ν with respect to σ.

2.3 Definition. By a *Fatou-filter mapping* we mean a function $\mathcal{G} : x \mapsto \mathcal{G}_x$ associating with each point $x \in X$ a filter base $\mathcal{G}_x$ of subsets of W so that for each $\nu \in M$, the ratio h_ν/h_σ has limit $d\nu/d\sigma(x)$ along $\mathcal{G}_x$ (i.e., along the filter generated by $\mathcal{G}_x$) for σ-a.e. $x \in X$. That is,

$$\lim_{\mathcal{G}_x} \frac{h_\nu}{h_\sigma} = \frac{d\nu}{d\sigma}(x) \quad \sigma\text{-a.e.}$$

2.4 Remark. Given a filter mapping initially defined only σ-a.e. on X, we will assume that the mapping has been extend to the rest of X with the filter containing only the set W.

2.5 Examples. For harmonic functions on the unit disk D with the integral representation of Example 2.1, the radial approach, nontangential approach, and minimal fine approach all provide examples of Fatou-filter mappings. That is, these are all approaches to the boundary for which the Fatou Boundary Limit Theorem holds.

In what follows, we write $\{h_\nu < h_\sigma\}$ to denote the set $\{w \in W : h_\nu(w) < h_\sigma(w)\}$. We will write $\sigma(A) = 0$ if this is true for A in the σ-completion of $\mathcal{B}$. We say that a Fatou-filter mapping $\mathcal{F}$ is coarser than a Fatou-filter mapping $\mathcal{G}$ if at each $x \in X$ the filter generated by $\mathcal{F}_x$ is contained in the filter generated by $\mathcal{G}_x$; we do not require the containment to be strict.

3. Zero sets

Our result is given in terms of a mapping $Z : \nu \mapsto Z_\nu$ from M into the σ-completion of $\mathcal{B}$. We call Z_ν, $\nu \in M$, the *zero set* for ν, and we think of it as the set of required zeros of $d\nu/d\sigma$. We assume that the following conditions hold for all ν, $\mu \in M$, all $E \in \mathcal{B}$, and all $\alpha > 0$ in $\mathbb{R}$:

$$\text{i) } Z_\nu \cap Z_\mu \subseteq Z_{\nu+\mu}, \quad \text{ii) } Z_{\alpha\nu} = Z_\nu, \quad \text{iii) } Z_\nu = \emptyset \text{ if } \sigma \leq \nu, \quad \text{iv) } Z_0 = X,$$

$$\text{v) If } \nu(E) = 0, \text{ then } \sigma(E \setminus Z_\nu) = 0.$$

3.1 Examples. The above conditions will hold if each Z_ν is the set where a limit 0 is obtained when computing a representative for $d\nu/d\sigma$ using a differentiation basis and the Lebesgue Differentiation Theorem (see [3]). Here, for each nonzero $\nu \in M$, we associate a positive value at each point where the limit does not exist and at each point outside of the support of σ. We can also choose the zero sets by using a Fatou-filter mapping on X; for each $\nu \in M$ we take the lim sup of the ratio h_ν/h_σ along the filter at each $x \in X$ and put $x \in Z_\nu$ if the result is 0. Since at least W is in the filter at each point of X, Condition iv is satisfied. These examples illustrate how, in practice, the selection of a zero set Z_ν takes into account the singular part of ν.

Now, in terms of the choice of M, σ, and the mapping Z, we construct a best, i.e., a coarsest, Fatou-filter mapping. For each $x \in X$ we set

$$\mathcal{F}_x := \{\{h_\nu < h_\sigma\} : \nu \in M,\ x \in Z_\nu\}.$$

3.2 Theorem. *The mapping $\mathcal{F} : x \mapsto \mathcal{F}_x$ is a Fatou-filter mapping on X. It is the coarsest such mapping with the property that for each $\nu \in M$,*

$$Z_\nu \subseteq \{x \in X : \lim_{\mathcal{F}_x} \frac{h_\nu}{h_\sigma} = 0\}.$$

If $y \in X$, $\nu \in M$, and h_ν/h_σ has limit 0 along $\mathcal{F}_y$, then $\{h_\nu < h_\sigma\}$ is in the filter generated by $\mathcal{F}_y$.

Proof. First, fix $x \in X$. To show that $\mathcal{F}_x$ is a filter base, we note that by Condition iii, $\emptyset \notin \mathcal{F}_x$; if $\{h_\nu < h_\sigma\} \in \mathcal{F}_x$ and $\{h_\mu < h_\sigma\} \in \mathcal{F}_x$, then by Condition i, $\{h_{\nu+\mu} < h_\sigma\} \in \mathcal{F}_x$, and of course

$$\{h_{\nu+\mu} < h_\sigma\} \subseteq \{h_\nu < h_\sigma\} \cap \{h_\mu < h_\sigma\}.$$

To show that the mapping $\mathcal{F}$ is a Fatou-filter mapping on X, we fix a set $E \in \mathcal{B}$ and a measure $\nu \in M$ with $\nu(E) = 0$. Since $\sigma(E \setminus Z_\nu) = 0$, we have $\{h_\nu < h_\sigma\} \in \mathcal{F}_x$ for σ-a.e. $x \in E$. By Theorem 1.1 of [3] we are done. (Here, we are using the mappings $\nu \mapsto h_\nu(w)$, $w \in W$, for the appropriate functionals on the elements $\nu \in M$; also see [2].) Fix $\nu \in M$. If $x \in Z_\nu$, then by Condition ii, for each natural number n, $\{nh_\nu < h_\sigma\} \in \mathcal{F}_x$, so the ratio h_ν/h_σ has limit 0 along $\mathcal{F}_x$. If $\mathcal{G}_x$ is also a filter of subsets of W along which h_ν/h_σ has limit 0, then $\{h_\nu < h_\sigma\} \in \mathcal{G}_x$. This shows that the mapping $x \mapsto \mathcal{F}_x$ is the coarsest Fatou-filter mapping on X yielding the desired zeros. If $y \in X$ and h_ν/h_σ has limit 0 along $\mathcal{F}_y$, then there is a set $F \in \mathcal{F}_y$ on which $h_\nu/h_\sigma < 1$, and so the set $\{h_\nu < h_\sigma\}$ is in the filter generated by $\mathcal{F}_y$. □

3.3 Remarks. The last statement of Theorem 3.2 shows that producing the filter map $x \mapsto \mathcal{F}_x$ is an idempotent operation. Once the sets of zeros Z_ν, $\nu \in M$, are produced by the map in the sense of Example 3.1, they do not change if the process is repeated .

Suppose the sets of zeros Z_ν, $\nu \in M$, are produced by taking limits with respect to a Fatou-filter map $x \mapsto \mathcal{G}_x$ such that for every potential p on W, the ratio p/h_σ has a limit 0 along $\mathcal{G}_x$ at σ-a.e. $x \in X$. Then for each potential p on W and every $x \in X$ where the limit of p/h_σ is 0, one can add the set $\{p < h_\sigma\}$ to the filter base $\mathcal{F}_x$ and generate a new filter base since $\mathcal{F}_x$ is a subset of the filter generated by $\mathcal{G}_x$ and $\{p < h_\sigma\}$ is an element of that filter. With this change, $\mathcal{F}$ is still a Fatou-filter mapping, and for each potential p, the ratio p/h_σ will have limit 0 along $\mathcal{F}_x$ at σ-a.e. $x \in X$. In other words, the Ratio Fatou Limit Theorem given in terms of superharmonic functions holds for $\mathcal{F}$ (see [2]).

4. Comparison with minimal fine and ordinary neighborhood filters

Starting with solutions of the Laplace equation on a domain in n-space, $n \geq 2$, suppose that X is the Martin boundary or the boundary produced in [7], that M is the set of all finite Borel measures on the minimal points of X, and that $h_\sigma \equiv 1$ on W. Let $\mathcal{G}_x$ be the minimal fine filter at each minimal point $x \in X$ and $\mathcal{G}_x = \{W\}$ at each non-minimal point $x \in X$. Producing zero sets as in Example 3.1 with the Fatou-filter mapping $\mathcal{G}$ (see [2]) and applying Theorem 3.2 yields a filter at each minimal point strictly coarser, i.e., strictly contained in, the minimal fine filter. This follows from the fact that any countable dense set is thin at each minimal point and the sets $\{h_\nu < h_\sigma\}$, $\nu \in M$, are open subsets of W. The following general result is also applicable at each point of X if X contains at least two points and is regular for the Dirichlet problem.

In what follows, we write $\mathcal{N}_x$ for the collection of all open sets U in the compact space $W \cup X$ with $x \in U$.

4.1 Theorem. *Suppose in the setting of Theorem 3.2 that* $h_\sigma \equiv 1$ *on* W*. Fix* $x \in X$ *and assume that for each* $U \in \mathcal{N}_x$ *there is a measure* $\nu \in M$ *with* $x \in Z_\nu$ *and* $\inf\{h_\nu(w) : w \in W \setminus U\} > 0$*. Then the filter generated by* $\mathcal{F}_x$ *refines the ordinary neighborhood filter; i. e., it contains the trace on* W *of each* $U \in \mathcal{N}_x$*.*

Proof. Fix $U \in \mathcal{N}_x$ and a corresponding $\nu \in M$ given by the hypothesis. Then for some $\alpha > 0$, the nonempty set

$$\{\alpha h_\nu < 1\} = \{h_{\alpha\nu} < 1\} \subset U. \qquad \square$$

5. Best filters and Doob's result

Finally, we return to Example 2.1, i.e., harmonic functions on the unit disk D with σ equal to normalized Lebesgue measure on the boundary C. Suppose that ν is a finite measure on C constructed from a sequence of point masses with the points converging to 1 but with 1 a point of symmetric density 0 for ν. Then the radial and nontangential limit at 1 of h_ν is 0. This is shown for the radial limit, in Theorem 3.1 of [6], and the result for the nontangential limit follows from Harnack's inequality using the argument at the end of the proof of Theorem 3.9 of [6]. (In general, the nontangental Fatou Boundary Limit Theorem for the disk can be proved using Theorem 1.1 of [3] and an argument, similar to what we have just given, which shows that $\lim h_\mu = 0$ for σ-almost every point of a Borel set E when μ is a measure with $\mu(E) = 0$.)

Let M_c consist of all measures with continuous densities on C. Define a filter mapping $\mathcal{G}$ on C by setting $\mathcal{G}_x = \{U \cap D : U \in \mathcal{N}_x\}$ for each $x \in C$. Then $\mathcal{G}$ is a Fatou-filter mapping for the convex cone generated by M_c and ν, but h_ν does not have limit 0 along $\mathcal{G}_1$. Of course, by adding sets to $\mathcal{G}_1$ of the form $\{h_{\alpha\nu} < 1\}$, $\alpha > 0$, we will generate a filter along which h_ν does converge to 0 at 1. With a similar modification of the filter $\mathcal{G}_x$ at each point $x = e^{i\theta}$ we can obtain a limit 0 at x for the measure ν rotated by the angle θ. These modifications are not needed, however, if we only want to obtain a limit 0 for h_ν at σ-a.e. point of C. This example demonstrates, in a simplified setting, the difference between Doob's result [4] and Theorem 3.2: There need not be a coarsest Fatou-filter map for all of M if the zeros are not specified.

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