AN OPTIMIZATION OF THE BESICOVITCH COVERING

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Abstract. Given an appropriate covering by balls of a set in a metric space, we construct an optimized version of the subcovering used in the proof of Besicovitch's theorem. The proof is nonstandard, and suggests a general method for optimizing standard geometric constructions.

In this note, we construct an optimized version of the covering by balls used in the proof of the Besicovitch covering theorem ([1]; also see [3]). That is, given an arbitrary set A in an appropriate metric space and a covering $\{B(a, r(a)) : a \in A\}$ by closed balls with bounded radii, we want to extract a subcovering with no center in the interior of any other ball of the subcovering and each point $a \in A$ in a ball with radius no smaller than $r(a)$. Easily constructed examples on the real line show that in the absence of additional assumptions on the radius function $r(\cdot)$, we must adjoin limit balls to the original covering. The proof that the desired subcovering then exists is nonstandard. David Berg has shown that a standard diagonalization argument works for σ-compact spaces. The proof given here uses the standard part map after replacing a magnification factor in the standard Besicovitch covering with a factor infinitely close to 1. This suggests a general method for optimizing standard geometric constructions.

Our setting is a metric space (X, ρ); we write $B(c, r)$ to denote a closed metric ball $\{x \in X : \rho(c, x) \leq r\}$ and $\mathrm{Cl}(E)$ to denote the closure of a set E in X. We will assume a local compactness condition which is trivially satisfied if closed balls in X are compact.

Theorem. *Let A be an arbitrary subset of X. Assume that at each point $a \in A$ there is centered a closed ball $B(a, r(a))$ with positive radius so that $\sup_{a\in A} r(a) < +\infty$. Also assume that for each $c \in A$, the set*

$$K(c) = \mathrm{Cl}\big(\big\{a \in A : c \in B\big(a, r(a)\big)\big\}\big)$$

is compact. With each point $p \in \mathrm{Cl}(A)$ we associate the set

$$L(p) = \{t > 0 : \forall \varepsilon > 0, \exists a \in A \ \text{ with } \ \rho(p, a) < \varepsilon \ \text{ and } \ |t - r(a)| < \varepsilon\}.$$

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There exists a collection of closed balls $\mathcal{B}$ with

$$\mathcal{B} \subseteq \{B(p,t) : p \in \mathrm{Cl}(A) \ \text{ and } \ t \in L(p)\}$$

such that no center of any ball in $\mathcal{B}$ is in the interior of any other ball in $\mathcal{B}$, and each $a \in A$ is in a ball $B(p,t) \in \mathcal{B}$ with $t \geq r(a)$. If the radius function r is continuous on A and $\lim_{a\to p} r(a) = 0$ at every point $p \in \mathrm{Cl}(A) \setminus A$, then $\mathcal{B} \subseteq \{B(a, r(a)) : a \in A\}$.

Proof. Given a real number $s > 1$, one may choose a point $a_1 \in A$ so that for each $a \in A$, $s \cdot r(a_1) > r(a)$. Setting $A_1 = A \setminus B(a_1, r(a_1))$, there is an $a_2 \in A_1$ so that for all $a \in A_1$, $s \cdot r(a_2) > r(a)$. By induction over the ordinals (see for example [3]), one may continue in this way to obtain a well ordered subset $\{a_\alpha\}$ of A such that $A \subseteq \cup_\alpha B(a_\alpha, r(a_\alpha))$. This is a Besicovitch covering of A. We will employ the transfer principle to obtain a set with the properties of $\{a_\alpha\}$ in a nonstandard structure containing X (see [2]). Fix $s = 1 + \varepsilon$ where ε is a positive infinitesimal. There is an internal subset $A_0 = \{a_\alpha\}$ of ${}^*A$ with an internal well ordering $\preceq$ such that for $r_\alpha = {}^*r(a_\alpha)$ we have ${}^*A \subseteq \cup_\alpha B(a_\alpha, r_\alpha)$, and when $a_\alpha \precneqq a_\beta$, $a_\beta \notin B(a_\alpha, r_\alpha)$ and $s \cdot r_\alpha > r_\beta$. Moreover, if a is in ${}^*A$ and a_α is the first element of A_0 with $a \in B(a_\alpha, r_\alpha)$, then $s \cdot r_\alpha > {}^*r(a)$. Suppose $a_\alpha \in A_0$ and $B(a_\alpha, r_\alpha)$ contains a standard point $c \in A$. Then a_α is in the nonstandard extension of the compact set $K(c)$, so a_α is infinitely close in terms of the internal metric ${}^*\rho$ to a standard point ${}^\circ a_\alpha \in K(c) \subseteq \mathrm{Cl}(A)$. If the standard part ${}^\circ r_\alpha$ of r_α is strictly positive, then ${}^\circ r_\alpha$ is in $L({}^\circ a_\alpha)$. Set

$$\mathcal{B} = \{B({}^\circ a_\alpha, {}^\circ r_\alpha) : a_\alpha \in A_0, \ {}^\circ r_\alpha > 0 \ \text{ and } \ \exists c \in A \ \text{ with } \ c \in B(a_\alpha, r_\alpha)\}.$$

To see this is a covering of A with the desired properties, fix balls $B({}^\circ a_\alpha, {}^\circ r_\alpha)$ and $B({}^\circ a_\beta, {}^\circ r_\beta)$ in $\mathcal{B}$ with $a_\alpha \precneqq a_\beta$. Then $a_\beta \notin B(a_\alpha, r_\alpha)$ and $(1+\varepsilon) \cdot r_\alpha \geq r_\beta$, so

$$\rho({}^\circ a_\beta, {}^\circ a_\alpha) \geq {}^\circ r_\alpha \geq {}^\circ r_\beta > 0.$$

If $c \in A$, and a_α is the first element of A_0 with $c \in B(a_\alpha, r_\alpha)$, then $\rho(c, {}^\circ a_\alpha) \leq {}^\circ r_\alpha$, so $c \in B({}^\circ a_\alpha, {}^\circ r_\alpha)$, and $(1+\varepsilon) \cdot r_\alpha > r(c)$, so ${}^\circ r_\alpha \geq r(c) > 0$. $\square$

There may be some redundancy in the collection $\mathcal{B}$. We can, of course, remove a ball $B({}^\circ a_\beta, {}^\circ r_\beta)$ when $B({}^\circ a_\beta, {}^\circ r_\beta) \subseteq \cup\{B({}^\circ a_\alpha, {}^\circ r_\alpha) \in \mathcal{B} : a_\alpha \precneqq a_\beta\}$. The proof given here illustrates the use of the transfer principle in optimizing a standard construction. Hermann Render has noted that the proof can also begin by fixing a hyperfinite set A_f containing every standard point in A, and successively selecting a ball with maximum radius and center in A_f from balls with centers not yet covered in A_f.

References

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