An intuitive approach to the Martin boundary

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Abstract

An intuitive probabilistic alternative for the construction of the Martin boundary is presented along with a construction of maximal representing measures for positive harmonic functions. © 2020 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.

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1. Introduction

Robert Martin’s 1941 boundary construction in [13] is now a fundamental tool in potential theory and probability theory. (See [2,3].) We suggest here an intuitive probabilistic alternative to Martin’s Green’s function construction. It begins with the author’s result with M. Insall and M. Marcinisiak in [7] showing that a compactifying boundary is formed by equivalence classes of points not infinitely close to standard points in the nonstandard extension of a metric (or even a regular) space. An extension in [6] shows that any Hausdorff compactification can be formed in this way. This raises the following question: What is an equivalence relation using probability theory for Martin’s boundary?1

The equivalence relation presented here produces a compactification that coincides with Martin’s for important examples. We also show, using [9], that the resulting construction leads to the correct representing measures for positive harmonic functions. This approach “looking inside” a domain only makes sense when one can speak of points that are neither points of an existing boundary nor points in a compact subset of the domain.

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2. General compactifications

In this section, we review joint work with Insall and Marciniak in [6,7]. We use basic concepts of Robinson’s [14] nonstandard analysis that are discussed in Appendix C of [11] and more deeply in the beginning of [12]. While the results of this section extend to a regular topological space, we restrict our attention here to a noncompact metric space \((W, d)\). A compactification of \(W\) is a compact space containing \(W\) as dense subspace.

Recall that if \(r\) is a nonstandard real number a finite distance from 0, then the standard part of \(r\) is denoted by \(\text{st}\, r\). It is the unique standard real number infinitely close to \(r\). Also, we let \(*\mathbb{N}_\infty\) denote the set of unlimited natural numbers, i.e., \(*\mathbb{N}\setminus\mathbb{N}\).

Fix a nonstandard extension \(*W\) of \(W\). A point \(y \in *W\) is called near-standard if there is a standard point \(x \in W\) with \(d(y, x) \asymp 0\). That is, the distance from \(x\) to \(y\) is infinitesimal. If \(y \in *W\) is not near-standard, then we say that \(y\) is remote in \(*W\). For example, the remote points in the nonstandard extension of the open unit disk are the points with distance from the origin infinitely close to 1. By the main result in [7] (summarized in Chapter 5 of [12]), given an equivalence relation on the remote points of \(*W\), the equivalence classes form the boundary points of a compactification of \(W\). By [6], every Hausdorff compactification can be formed this way. The following example extends even to topological spaces that are just regular.

**Example 2.1 (Stone–Čech).** Let \(\mathcal{F}\) be the set of all bounded continuous real-valued functions on \(W\). The Stone–Čech compactification is produced by the equivalence relation that sets remote points \(x\) and \(y\) equivalent if and only if for all \(f\) in \(\mathcal{F}\),

\[ *f(x) - *f(y) \asymp 0. \]

3. A probabilistic equivalence relation

Let \(W\) be an open connected domain in Euclidean space or manifold suitable for potential theory and Brownian motion. Let \(n \mapsto K_n, n \in \mathbb{N}\), be a compact exhaustion of \(W\). That is, each \(K_n\) is a compact set contained in the interior of \(K_{n+1}\), and \(\cup_{n \in \mathbb{N}} K_n = W\). We may assume that each \(K_n\) is the closure of a connected region that is regular for the Dirichlet problem. By Robinson’s compactness criterion, a point \(x\) is remote in \(*W\) if and only if it is outside \(*K_n\) for every \(n \in \mathbb{N}\).

For all \(n \in \mathbb{N}\) and \(z \in W \setminus K_n\), let \(\rho^n_z\) be the conditional probability measure on \(\partial K_n\) given by the exit distribution of a Brownian particle for \(W \setminus K_n\). That is, \(\rho^n_z\) is the restriction to \(\partial K_n\) of harmonic measure for \(W \setminus K_n\) but normalized to total mass 1. The function \(z \mapsto \rho^n_z\) extends to the points of \(*W \setminus *K_n\).

Given remote points \(x\) and \(y\), we set

\[ f_n(x, y) := \text{st} \left( \left| *\rho^n_x - *\rho^n_y \right| (*\partial K_n) \right). \]

Note that \(f_n(x, y)\) is a real-valued sequence. We call remote points \(x\) and \(y\) equivalent, and write \(x \sim y\), if that sequence has limit 0. That is,

\[ x \sim y \iff \lim_{n \to \infty} f_n(x, y) = 0. \]

There is still a great deal of freedom in choosing an exhaustion \(n \mapsto K_n\).
4. Preserving an existing boundary

A topological boundary of $W$ may be reproduced by the above equivalence relation. Recall, for example, the celebrated result of Hunt and Wheeden [5], showing that the topological boundary of a Lipschitz domain is the Martin boundary. In any case, each remote point $x$ will be infinitely close to a unique standard point $stx$ in the topological boundary. The following condition is clear.

**Proposition 4.1.** A topological boundary of $W$ will be produced by the equivalence relation

$$x \sim y \iff stx = sty.$$

**Example 4.2 (Unit Disk).** For each $n \in \mathbb{N}$, let $K_n$ be the closed disk of radius $1 - 1/n$ centered at the origin. Two remote points $x$ and $y$ each have a standard part on the unit circle, and $x \sim y$ if and only if $stx = sty$. Therefore, the corresponding compactification is the closed unit disk.

**Example 4.3 (Cut Unit Disk).** Let $W$ be the open unit disk from which the interval $(0, 1)$ in the $x$-axis has been removed. Given a positive $\varepsilon \simeq 0$, the remote points $1/2 + si$ and $1/2 - si$ have the same standard part in the complex plane, but they are not equivalent. The equivalence relation replaces the part of the topological boundary formed by interval $(0, 1)$ on the $x$-axis with two similar but distinct intervals.

5. Two exotic examples

Here are two more examples where the equivalence relation produces the Martin boundary.

**Example 5.1.** Let $S$ be the topological boundary of the sphere in $\mathbb{R}^3$ of radius 2 centered at the point $(0, 0, 2)$. Let $T$ be the solid sphere of radius 1 centered at $(0, 0, 1)$. Clearly, $T$ is contained inside $S$, and the boundary of $T$ intersects $S$ just at the origin. Let $W$ be the open region between $T$ and $S$. As compact sets $K_n$ fill $W$, they must surround much of $T$ above the origin. We may assume that the bottom boundaries of sets $K_n$ for large $n$ are annular 2-dimensional regions. It follows that the boundary produced by the equivalence relation replaces the origin in the topological boundary of $W$ with a ring of points. This coincides with the Martin boundary for $W$.

**Example 5.2.** Start with the open square in $\mathbb{R}^2$ given by $0 < x < 1, 0 < y < 1$. Assume that for each $n \in \mathbb{N}$, the following interval has been removed from the square.

$$x = 1/n, \quad 0 < y < 1 - 1/n.$$

Let $W$ be the resulting region. Suppose $(\xi, \eta)$ is a remote point in $^*W$ with $\xi$ strictly between $1/\omega$ and $1/(\omega + 1)$, where $\omega$ is in $^*\mathbb{N}_{\infty}$. Then the path of any Brownian particle starting at that point that then exits $W$ from the boundary of a standard compact set $K_n$ must contain points for which the $x$-coordinate is infinitesimal and the $y$ coordinate is infinitely close to 1. It follows that the boundary produced by the equivalence relation intersects the $y$-axis at the point $(0, 1)$. This coincides with the Martin boundary for $W$. 
6. Representing measures

Fix \( x_0 \in W \). Let \( \mathcal{H}^1 \) be space of all positive harmonic functions \( h \) on \( W \) with \( h(x_0) = 1 \). The set \( \mathcal{H}^1 \) is convex and compact with respect to the topology of uniform convergence on compact subsets of \( W \), i.e., the ucc topology. Each \( h \in \mathcal{H}^1 \) is represented by a unique probability measure on the extreme elements of \( \mathcal{H}^1 \). We will use the following construction of that measure from [9]. It is an early application, discussed at the 1974 Oberwolfach Conferences on Potential Theory, of the general measure construction later published in [8], with many other applications, such as [1]. A standard, weak-limit construction of representing measures, established using the nonstandard construction, can be found in [10].

Fix \( \gamma \in \*\mathbb{N}_{\infty} \). Let \( U \) denote the internal interior of \( K_\gamma \); let \( C \) denote \( \partial K_\gamma \). Recall that \( C \) is an internal Dirichlet regular boundary of \( U \). For each \( x \in U \), let \( \mu_x \) denote the internal harmonic measure for \( x \) on \( C \). That is, for each \( x \in U \) and each internally continuous \( g \) on \( C \), the map

\[
x \mapsto \int_C g(s)d\mu_x(s)
\]

gives the value at \( x \) of the internal harmonic extension of \( g \) from \( C \) to \( U \).

We next fix a hyperfinite partition of \( C \) consisting of internal Borel subsets. We assume the partition is so fine that for every \( h \in \mathcal{H}^1 \), \( \*h \) has infinitesimal variation on each set in the partition. We denote by \( \{A_i\} \) an internal indexed subfamily of the partition such that for each index \( i \), \( \mu_{x_0}(A_i) > 0 \), and \( \mu_{x_0}(C \setminus \cup_i A_i) = 0 \), whence for each \( x \in U \), \( \mu_x(C \setminus \cup_i A_i) = 0 \).

For each index \( i \), fix \( y_i \in A_i \). The \( \*\mathbb{R} \)-valued map on \( U \) given by

\[
z \mapsto \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)}
\]

is an internal harmonic function in \( U \). It is the solution for the function that is 1 on \( A_i \) and 0 on the rest of \( C \), then normalized to be 1 at \( x_0 \). The real-valued function

\[
x \mapsto \left( \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)} \right)_{x \in W}
\]

is its standard part in \( \mathcal{H}^1 \) with respect to the ucc topology.

Given any standard \( h \in \mathcal{H}^1 \) and \( x \in W \),

\[
h(x) = \int_C \*h(y)d\mu_x(y) \\
    \simeq \sum_i \*h(y_i)\mu_x(A_i) \\
    = \sum_i \*h(y_i)\mu_{x_0}(A_i)\frac{\mu_x(A_i)}{\mu_{x_0}(A_i)}.
\]

The hyperfinite set of weights \( \{\mu_x(A_i)\}/\mu_{x_0}(A_i) \) forms an internal measure \( \varphi_h \) on the indexed set \( \left\{ \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)} \right\} \) of internal harmonic functions on \( U \). That is, for each \( i \), the function

\[
z \mapsto \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)}, \quad z \in U
\]
is given the weight \( \psi(\lambda)\mu_\lambda(A_\lambda) \). Using the construction in [8], the measure \( \psi_\lambda \) is converted to a standard measure \( L(\varphi_\lambda) \), still on the same set of internally harmonic functions. With an early but specific use of the measurability of the standard part map applied to the mapping

\[
\frac{\mu(A_\lambda)}{\mu_\lambda(A_\lambda)} \mapsto \text{uccst} \left( \frac{\mu(A_\lambda)}{\mu_\lambda(A_\lambda)} \right),
\]

the now standard measure \( L(\varphi_\lambda) \) is moved to a probability measure \( P_\lambda \) on \( \mathcal{H}_1 \).

For each \( h \in \mathcal{H}_1 \), for each \( x \in W \),

\[
h(x) = \int_{g \in \mathcal{H}_1} g(x) dP_\lambda(g).
\]

That is, \( P_\lambda \) is a representing measure for \( h \). If \( h \) equals an affine combination \( \sum_j \alpha_j h_j \) of functions in \( \mathcal{H}_1 \), then \( P_\lambda = \sum_j \alpha_j P_\lambda \). A consequence of this fact, communicated to the author by B. Fuchssteiner, is that the Fuchssteiner corollary in [4] of a theorem of Cartier, Fell, and Meyer (see [9]), shows that \( P_\lambda \) is the unique representing measure for \( h \) on the extreme points of \( \mathcal{H}_1 \).

The problem remains to connect this construction of representing measures with the boundary \( \partial W \) formed using our equivalence relation. For each \( h \in \mathcal{H}_1 \), the internal measure we have constructed can also be formed on \( C \) using the weights \( \psi(\lambda)\mu_\lambda(A_\lambda) \). Using [8], that measure can be transformed into a standard probability measure on \( C \), and then moved to \( \partial W \) using the standard part map. It is more important, however, to map \( \partial W \) into \( \mathcal{H}_1 \).

For each \( z \in \partial W \), let \( C_z \) denote the set of remote points in the equivalence class forming \( z \) but also in \( C \). Every connected standard neighborhood of \( z \) contains points of \( W \), so the nonstandard extension contains points of \( C \). It follows that \( C_z \) is not empty. If \( y \in C_z \) is in \( A_i \) for some \( i \), then it is associated with the harmonic function

\[
\text{uccst} \left( \frac{\mu(A_i)}{\mu_\lambda(A_i)} \right) \in \mathcal{H}_1.
\]

If \( y \in C_z \) is not in any \( A_i \), we associate no function with \( y \).

We now assume that for each \( z \in \partial W \), we associate the same element of \( \mathcal{H}_1 \) with each \( y \in C_z \), for which we have associated a function. This forms a map from all or part of \( \partial W \) into \( \mathcal{H}_1 \), and the representing measure on \( \mathcal{H}_1 \) can then be viewed as a measure on \( \partial W \).

References
