Purification of Measure-Valued Maps

Peter Loeb and Yeneng Sun

Abstract. Given a measurable mapping f from a nonatomic Loeb probability space $(T, \mathcal{T}, P)$ to the space of Borel probability measures on a compact metric space A, we show the existence of a measurable mapping g from $(T, \mathcal{T}, P)$ to A itself such that f and g yield the same values for the integrals associated with a countable class of functions on $T \times A$. A corollary generalizes the classical result of Dvoretzky-Wald-Wolfowitz on purification of measure-valued maps with respect to a finite target space; the generalization holds when the domain is a nonatomic, vector-valued Loeb measure space and the target is a complete, separable metric space. A counterexample shows that the generalized result fails even for simple cases when the restriction of Loeb measures is removed. As an application, we obtain a strong purification for every mixed strategy profile in finite-player games with compact action spaces and diffuse and conditionally independent information.

1. Introduction

In 1951, Dvoretzky, Wald and Wolfowitz used the Lyapunov theorem for vector measures to establish the following result in [**9**, Theorem 4] (also announced in [**8**, Theorem 1] and in [**10**, Theorem 2.1]).

Theorem 1.1. *Let A be a finite set, $(T, \mathcal{T})$ a measurable space, and μ_k, $k = 1, \cdots, m$, finite, nonatomic signed measures on $(T, \mathcal{T})$. Let f be a mapping from T to the space $\mathcal{M}(A)$ of probability measures on A such that for each $a \in A$, $f(\cdot)(\{a\})$ is $\mathcal{T}$-measurable. Then there exists a $\mathcal{T}$-measurable function g from T to A such that for each $a \in A$,*

$$\int_T f(t)(\{a\})\mu_k(dt) = \mu_k(\{t \in T : g(t) = a\}).$$

This theorem justifies the elimination, i.e., purification, of randomness in various settings. In games, for example, T represents the space of information available to the game's players, and A represents the set of actions players may choose, given the available information $t \in T$.

Each player's objective is to maximize their own expected payoff, which depends not only on that player's choice of action but also on that of all the other players. (Our use of "their" is consistent with the increasing use of some form of *they* with singular, generic antecedents that has its origins in the fourteenth century; see [**4**].) For each player, a mapping from the space of information T to particular actions in A is called a pure strategy. If the mapping is not to A itself but to the space $\mathcal{M}(A)$ of probability measures on A, then that mapping is called a mixed strategy; here the player chooses a "lottery on A". A Nash equilibrium is achieved when every player is satisfied with their choice of strategy given the choices of all the other players. In quite general settings, such an equilibrium can be achieved when the players choose a mixed strategy. In the more restrictive settings where Theorem 1.1 or an extension applies, those strategies can then be purified to obtain an equilibrium with the same expected payoff for all the players.

In fact, Theorem 1.1 was applied by Dvoretzky, Wald and Wolfowitz to the purification of both statistical decision procedures (see [**8**, Theorems 5 and 6], [**10**, Theorems 3.1 and 3.2, Section 4, Theorems 5.1 and 5.2]), and of mixed strategies in two-person zero-sum games with finite action sets (see [**8**, Theorems 2 and 3], [**10**, Section 9] on two-person zero-sum games). The relevance of Theorem 1.1 to the purification problem in finite games with finite action spaces and incomplete and diffuse information was already suggested in [**20**, Footnote 3] and in [**19**, Section 5]. A unified approach to purification problems in finite-action games using Theorem 1.1 is presented in [**14**].

Theorem 1.1 and the applications just noted are restricted to the case of a finite action space A. We will remove that restriction by establishing a result valid for a compact metric space and even a complete separable metric space. Even when A is a closed, finite interval in the real line, however, Example 2.7 below shows that there is no extension of Theorem 1.1 when T is the unit interval supplied with Lebesgue measure and another measure having a continuous density function. To obtain our extension, we require that T with its associated measures are nonatomic measure spaces of the kind introduced by the first author in [**17**], and now called "Loeb spaces" in the literature. Using such a space T, we will obtain a general extension of Theorem 1.1 and a corresponding application to games.

In Section 2, we consider the purification of measure-valued maps. Theorem 2.2 shows that for a measurable mapping f from a nonatomic Loeb probability space $(T, \mathcal{T}, P)$ to the space of Borel probability measures on a compact metric space A, one can find a measurable mapping g from $(T, \mathcal{T}, P)$ to A such that f and g yield the same values for the integrals associated with a countable class of functions on $T \times A$. Corollaries 2.4 and 2.6 then generalize Theorem 1.1 to the case of a compact metric space and a complete separable metric space. Corollary 2.4 is then applied in Section 3 to obtain in Theorem 3.2 the existence of a strong purification for every mixed strategy profile in finite-player games with compact action spaces and diffuse and conditionally independent information. The example in [**13**] also shows that such a purification result is no longer valid when a Loeb space is not used.

2. Main Theorem

Let $\mathbb{N}$ denote the natural numbers, $\mathbb{R}$ the reals, and $\mathbb{R}_+$ the nonnegative reals. For this section, we fix an $\aleph_1$-saturated extension of a standard superstructure containing at least the real numbers. In that extension, we let T be an internal set, $\mathcal{T}_0$ an internal algebra on T, and P_0 an internal, finitely additive set function from $(T, \mathcal{T}_0)$ to ${}^*\mathbb{R}_+$ with $P_0(T) = 1$. We let $(T, \mathcal{T}, P)$ be the Loeb probability space generated by $(T, \mathcal{T}_0, P_0)$. (See, for example, [**1**] or [**18**]) We assume that P is nonatomic. We will use st to denote the standard part operation, and write $a \simeq b$ when $a - b$ is infinitesimal in ${}^*\mathbb{R}$.

Let A be a compact metric space. We denote the collection of Borel subsets of A by $\mathcal{B}$, and we let $\mathcal{M}(A)$ be the space of Borel probability measures on A with the topology of weak convergence. For any mapping f from T to $\mathcal{M}(A)$, the $\mathcal{T}$-measurability of f with respect to this topology is equivalent to the $\mathcal{T}$-measurability of $f(\cdot)(B)$ for each $B \in \mathcal{B}$. The space of continuous real-valued functions on A is supplied with the sup-norm topology. For any $\gamma \in \mathcal{M}(A)$, $\operatorname{supp} \gamma$ is the support of γ, i.e., the complement of the union of all open γ-null subsets of A.

Let $\mathcal{F}$ be the collection of functions ϕ from $T \times A$ to $\mathbb{R}$ such that $\phi(\cdot, a)$ is $\mathcal{T}$-measurable on T for each $a \in A$ and $\phi(t, \cdot)$ is continuous on A for each $t \in T$; assume that for each $\phi \in \mathcal{F}$ there is a P-integrable function α_ϕ from T to $\mathbb{R}_+$ with $|\phi(t, a)| \leq \alpha_\phi(t)$ for all $(t, a) \in T \times A$.

By a uniform lifting of $\phi \in \mathcal{F}$ (with respect to the internal measure P_0), one means an internal function $\phi_0 : T \times {}^*A \to {}^*\mathbb{R}$ such that for each $a \in {}^*A$, $\phi_0(\cdot, a)$ is $\mathcal{T}_0$-measurable and for P-almost all $t \in T$, $\phi_0(t, a) \simeq \phi_0(t, \operatorname{st} a)$ holds for any $a \in {}^*A$. The existence of such uniform liftings follows from essentially the same proof as given by Keisler in [**12**] and generalized in Proposition 4.3.13 of [**1**].

Lemma 2.1. *Let $\mathcal{D}$ be a countable subcollection of $\mathcal{F}$. Assume that there is a sequence of $\mathcal{T}$-measurable mappings $\{g_n,\ n \in \mathbb{N}\}$ from T to A such that for each $\phi \in \mathcal{D}$, the sequence $\int_T \phi(t, g_n(t)) P(dt)$ converges; let $c_\phi \in \mathbb{R}$ denote the limit. Then, there is a $\mathcal{T}$-measurable mapping g*

from T to A such that for each $\phi \in \mathcal{D}$,

$$\int_T \phi(t, g(t))P(dt) = c_\phi. \tag{1}$$

PROOF. For each $n \in \mathbb{N}$, let $h_n : T \to {}^*A$ be a $\mathcal{T}_0$-measurable lifting of g_n with respect to the internal measure P_0. For each $\phi \in \mathcal{D}$, let $\phi_0 : T \times {}^*A \to {}^*\mathbb{R}_+$ be a uniform lifting of ϕ.

By our assumptions, for any $\phi \in \mathcal{D}$, $\phi_0(\cdot, h_n(\cdot))$ is a $\mathcal{T}_0$-measurable lifting of $\phi(\cdot, g_n(\cdot))$, whence

$$\int_T \phi(t, g_n(t))P(dt) \simeq \int_T \phi_0(t, h_n(t))P_0(dt),$$

and so

$$\lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} \left(\mathrm{st} \left| \int_T \phi_0(t, h_n(t))P_0(dt) - c_\phi \right| \right) = 0. \tag{2}$$

Using $\aleph_1$-saturation, we may extend the sequence h_n to an internal sequence and choose an unlimited integer $H \in {}^*\mathbb{N}$ so that for every $\phi \in \mathcal{D}$,

$$\int_T {}^*\phi_0(t, h_H(t))P_0(dt) \simeq c_\phi. \tag{3}$$

The desired function g is obtained by setting $g(t) := \mathrm{st}(h_H(t))$ at each $t \in T$, since then $\int_T \phi(t, g(t))P(dt) = c_\phi$ for each $\phi \in \mathcal{D}$. $\square$

THEOREM 2.2. *Let $\mathcal{D}$ be a countable subcollection of $\mathcal{F}$. Given a $\mathcal{T}$-measurable mapping f from T to $\mathcal{M}(A)$, there is a $\mathcal{T}$-measurable mapping g from T to A itself such that for each $\phi \in \mathcal{D}$,*

$$\int_T \int_A \phi(t, a)f(t)(da)P(dt) = \int_T \phi(t, g(t))P(dt). \tag{4}$$

PROOF. We will first prove the result for the case that $f : T \to \mathcal{M}(A)$ is simple. We let $\{S_j\}_{j=1}^N$ denote the corresponding $\mathcal{T}$-measurable partition of T such that f is identically equal to a measure $\gamma_j \in \mathcal{M}(A)$ on S_j. Now for any $\phi \in \mathcal{D}$,

$$\int_T \int_A \phi(t, a)f(t)(da)P(dt) = \sum_{j=1}^N \int_{S_j} \int_A \phi(t, a)\gamma_j(da)P(dt). \tag{5}$$

For each $m \geq 1$, fix a Borel measurable, finite partition $\mathcal{P}^m = \{A_1^m, \ldots, A_{k_m}^m\}$ of A such that the diameter of each set in $\mathcal{P}_m$ is at most $1/2^m$, and $\mathcal{P}_{m+1}$ is a refinement of $\mathcal{P}_m$. For each k with $1 \leq k \leq k_m$, pick a point a_k^m in A_k^m.

For each $\phi \in \mathcal{D}$ and positive integers m and k with $k \leq k_m$, define a real-valued, signed measure $\nu_k^{\phi,m}$ on $(T, \mathcal{T})$ by setting for each $S \in \mathcal{T}$

$$\nu_k^{\phi,m}(S) := \int_S \phi(t, a_k^m) P(dt). \tag{6}$$

Note that for each $S \in \mathcal{T}$,

$$\left| \nu_k^{\phi,m}(S) \right| \leq \int_T \alpha_\phi(t) P(dt) < +\infty.$$

Let ν be the vector measure on $(T, \mathcal{T})$ with values in $\mathbb{R}^{k_m}$ for which the k^{th} component is $\nu_k^{\phi,m}$. Since ν is nonatomic and $\sum_{k=1}^{k_m} \gamma_j(A_k^m) = 1$, it follows from the Lyapunov Theorem that each S_j can be decomposed by a finite, $\mathcal{T}$-measurable partition $\{T_1^{j,m}, \ldots, T_{k_m}^{j,m}\}$ so that for each k with $1 \leq k \leq k_m$,

$$\nu\left(T_k^{j,m}\right) = \gamma_j(A_k^m)\nu(S_j). \tag{7}$$

Taking components, this means that for each $\phi \in \mathcal{D}$, and all positive integers j, m and k with $j \leq N$ and $k \leq k_m$, we have

$$\nu_k^{\phi,m}\left(T_k^{j,m}\right) = \gamma_j(A_k^m)\nu_k^{\phi,m}(S_j); \tag{8}$$

that is,

$$\int_{T_k^{j,m}} \phi(t, a_k^m) P(dt) = \gamma_j(A_k^m) \int_{S_j} \phi(t, a_k^m) P(dt). \tag{9}$$

For each $m \geq 1$, define a $\mathcal{T}$-measurable mapping g_m from T to A so that for each $k \leq k_m$ and each $j \leq N$,

$$g_m(t) \equiv a_k^m \quad \text{on} \quad T_k^{j,m}. \tag{10}$$

Given $\phi \in \mathcal{D}$, it follows from Equation (9) that

$$\int_T \phi(t, g^m(t)) P(dt) = \sum_{j=1}^{N} \int_{S_j} \sum_{k=1}^{k_m} \phi(t, a_k^m)\gamma_j(A_k^m) P(dt). \tag{11}$$

Now for each $t \in T$, $\phi(t, \cdot)$ is continuous on A and $|\phi(t, \cdot)| \leq \alpha_\phi(t)$. Moreover, $\sum_{k=1}^{k_m} \phi(t, a_k^m)\gamma_j(A_k^m)$ is a Riemann-sum approximation to $\int_A \phi(t, a)\gamma_j(da)$, so

$$\lim_{m \to \infty} \sum_{k=1}^{k_m} \phi(t, a_k^m)\gamma_j(A_k^m) = \int_A \phi(t, a)\gamma_j(da). \tag{12}$$

By the Dominated Convergence Theorem, it follows from Equations (5), (11) and (12) that

(13)
$$\lim_{m\to\infty}\int_T \phi(t,g^m(t))P(dt) = \lim_{m\to\infty}\sum_{j=1}^{N}\int_{S_j}\sum_{k=1}^{k_m}\phi(t,a_k^m)\gamma_j(A_k^m)P(dt)$$

$$= \sum_{j=1}^{N}\int_{S_j}\int_A \phi(t,a)\gamma_j(da)P(dt) \tag{14}$$

$$= \int_T\int_A \phi(t,a)f(t)(da)P(dt). \tag{15}$$

Since this is true for each $\phi \in \mathcal{D}$, it follows from Lemma 2.1 that there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $\phi \in \mathcal{D}$, equation (4) holds.

We continue with the proof for an arbitrary measurable $f : T \to \mathcal{M}(A)$. Since $\mathcal{M}(A)$ is a compact metric space under the Prohorov metric ρ (which induces the topology of weak convergence of measures; see, for example, [**6**]), there is a sequence of simple functions $\{f^n\}_{n=1}^\infty$ from $(T,\mathcal{T})$ to $\mathcal{M}(A)$ such that

$$\forall t \in T,\ \lim_{n\to\infty}\rho(f^n(t),f(t)) = 0. \tag{16}$$

For each $\phi \in \mathcal{D}$ and $t \in T$, $\phi(t,\cdot)$ is continuous, and $f^n(t)$ converges to $f(t)$ in the topology of weak convergence of measures on $\mathcal{M}(A)$. Moreover,

$$\left|\int_A \phi(t,a)f^n(t)(da)\right| \le \alpha_\phi(t).$$

By the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_T\int_A \phi(t,a)f^n(t)(da)P(dt) = \int_T\int_A \phi(t,a)f(t)(da)P(dt). \tag{17}$$

Since for each $n \ge 1$, f^n is a simple function, there is a $\mathcal{T}$-measurable mapping g^n from T to A such that for each $\phi \in \mathcal{D}$,

$$\int_T\int_A \phi(t,a)f^n(t)(da)P(dt) = \int_T \phi(t,g^n(t))P(dt), \tag{18}$$

whence

$$\lim_{n\to\infty}\int_T \phi(t,g^n(t))P(dt) = \int_T\int_A \phi(t,a)f(t)(da)P(dt). \tag{19}$$

By Lemma 2.1, there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $\phi \in \mathcal{D}$, equation (4) holds for f. $\square$

Remark 2.3. An elementary proof by David Ross of Lyapunov's theorem can be found in [**21**]. Alternatively, one can use the first author's Lyapunov theorem [**16**] in proving Theorem 2.2, but then all of the simple functions must be modified on a P-null set T_0 so that for each of them, the corresponding partition sets S_j of T are internal. In this case, the limit in Equation (16) is for all $t \in T \backslash T_0$. The simple proof in [**16**] employs a theorem of Steinitz [**24**], which for our purposes says that for each $n \in \mathbb{N}$, there is a positive constant C_n such that for any collection of vectors from the unit ball of Euclidean space $\mathbb{R}^n$ with sum 0 there is an ordering for which all partial sums are within the closed ball of radius C_n. An easy proof of Bergström in a difficult to obtain article [**5**] uses induction on n: Clearly, 1 suffices for C_1. Given C_n and an indexed collection of vectors from the unit ball of $\mathbb{R}^{n+1}$ adding to 0, there is a subset I_1 of the index set with the sum of the corresponding vectors a vector V of maximum norm. The complimentary collection I_2 of indices gives a sum $-V$. Let H be the hyperplane through the origin perpendicular to the line L through 0, V, and $-V$. Since the projection onto H of the vectors indexed by I_1 add to 0, we may order them so that every partial sum of those projections is inside the closed ball of radius C_n in H. We may similarly order the vectors indexed by I_2. Since V has maximum norm, the inner product of each vector indexed by I_1 with V is positive, while the inner product of each vector indexed by I_2 with V is negative. Keeping the two orders in taking vectors from I_1 and I_2, we may order the vectors indexed by $I_1 \cup I_2$ so that every partial sum has a projection on L of length at most 1. It follows that $\sqrt{4C_n^2+1}$ suffices for C_{n+1}.

Corollary 2.4. *For each k in a finite or countably infinite set K, let μ_k be a finite signed measure on $(T, \mathcal{T})$ that is absolutely continuous with respect to P. For each j in a finite or countably infinite set J, let ψ_j be an element of $\mathcal{F}$. If f is a $\mathcal{T}$-measurable mapping from T to $\mathcal{M}(A)$, then there is a $\mathcal{T}$-measurable mapping g from T to A such that $g(t) \in \operatorname{supp} f(t)$ for P-almost all $t \in T$, and for all $k \in K$, $j \in J$, $B \in \mathcal{B}$, and bounded Borel measurable functions θ on A,*

(1) $\int_A \psi_j(t,a) f(t)(da) P(dt) = \int_T \psi_j(t, g(t)) P(dt)$,
(2) $\int_T f(t)(B) \mu_k(dt) = \mu_k\left(g^{-1}[B]\right)$,
(3) $\int_T \int_A \theta(a) f(t)(da) \mu_k(dt) = \int_T \theta(g(t)) \mu_k(dt)$.

Proof. Let d denote the metric on A, and let Φ be the function from $T \times A$ to $\mathbb{R}$ defined by setting $\Phi(t,a) := d(a, \operatorname{supp} f(t))$. From the definition of the support of $f(t)$, it is clear that for any open set O, $O \cap \operatorname{supp} f(t) \neq \emptyset$ is equivalent to $f(t)(O) > 0$, whence, by Theorem 14.78 in [**2**], $\Phi \in \mathcal{F}$. For each $k \in K$, we let β_k be the Radon-Nikodym derivative of μ_k with respect to P. Let $\mathcal{G}$ be a countable dense set in the space of continuous functions on A with the supnorm topology. Let $\mathcal{D} \subseteq \mathcal{F}$ consist of all the functions, ψ_j, $j \in J$, together with Φ and all the functions $\beta_k(t)h(a)$ on $T \times A$ for $h \in \mathcal{G}$ and $k \in K$. Now by Theorem 2.2, there is a $\mathcal{T}$-measurable mapping g from T to A such that for each $\phi \in \mathcal{D}$,

$$\int_T \int_A \phi(t,a) f(t)(da) P(dt) = \int_T \phi(t, g(t)) P(dt). \tag{20}$$

Since Equation (20) holds for $\phi = \Phi$,

$$\begin{aligned}&\int_T d(g(t), \operatorname{supp} f(t)) P(dt)\\ &= \int_T \int_{\operatorname{supp} f(t)} d(a, \operatorname{supp} f(t)) f(t)(da) P(dt) = 0.\end{aligned}$$

Therefore, $g(t) \in \operatorname{supp} f(t)$ for P-almost all $t \in T$.

Conclusion 1 follows from Equation (20) applied to the functions $\phi = \psi_j$, $j \in J$. Conclusion 2 is equivalent to Conclusion 3, and Conclusion 3 is a consequence of the fact that for each $k \in K$, Equation (20) holds for all the functions $\beta_k(t)h(a)$, $h \in \mathcal{G}$. That is, we know that

$$\int_T \int_A h(a) f(t)(da) \mu_k(dt) = \int_T h(g(t)) \mu_k(dt). \tag{21}$$

holds for all $h \in \mathcal{G}$, thus for all continuous functions on A, and therefore for all bounded Borel measurable functions on A. $\square$

Remark 2.5. For a $\mathcal{T}$-measurable mapping f from T to $\mathcal{M}(A)$, Theorem 3 in [**23**] shows that there is a $\mathcal{T}$-measurable mapping g from T to A such that $g(t) \in \operatorname{supp} f(t)$ for P-almost all $t \in T$, and

$$\forall B \in \mathcal{B}, \int_T f(t)(B) P(dt) = Pg^{-1}(B).$$

This is related to the present work as a special case of Corollary 2.4 that involves only the measure P. A more distant relationship exists with Cutland's work on control theory [**7**] in which he considers a probability-valued map on $[0,1]$ supplied with the usual differential system, and replaces that "relaxed control" with a point-valued control defined on a different, hyperfinite space.

The following generalization of the Dvoretzky-Wald-Wolfowitz theorem on purification in a finite target space is a consequence of Corollary 2.4 for the case that no function ψ_j is taken from $\mathcal{F}$; i.e., the index set J is empty. Moreover, for this generalization we can let

A be a complete separable metric space since there always exists a Borel bijection from such a space to a compact metric space. That is, if A is uncountable, then it follows from Kuratowski's theorem (see [**22**], p. 406) that there is a Borel bijection from A to $[0,1]$. On the other hand, for a countable set A, one can use a bijection from A to $\{0,1,1/2,\dots,1/n,\dots\}$. We also note that given any finite or countably infinite collection of finite, nonatomic, signed Loeb measures μ_k, one can always find an nonatomic Loeb probability measure P with respect to which they are all absolutely continuous. We fix the measurable space $(T,\mathcal{T})$ as before.

COROLLARY 2.6. *Let K be a finite or countably infinite set, and let A be a complete separable metric space. For each $k \in K$, let μ_k be a nonatomic, finite, signed Loeb measures on $(T,\mathcal{T})$. If f is a $\mathcal{T}$-measurable mapping from T to $\mathcal{M}(A)$, then there is a $\mathcal{T}$-measurable mapping g from T to A such that for all $k \in K$ and all Borel sets B in A, $\int_T f(t)(B)\mu_k(dt) = \mu_k\left(g^{-1}[B]\right)$. This is equivalent to the condition that for any bounded Borel measurable function θ on A,*

$$\int_T \int_A \theta(a) f(t)(da)\mu_k(dt) = \int_T \theta(g(t))\mu_k(dt). \tag{22}$$

The following example shows that Corollary 2.6 is false without the use of Loeb measures. Since Corollary 2.6 is a special case of Theorem 2.2 and also a special case of Corollary 2.4, each of those results fails without the use of Loeb measures.

EXAMPLE 2.7. Let $(T,\mathcal{T})$ be the unit interval with the Borel σ-algebra. Let $A = [-1,1]$ and $f(t) = (\delta_t + \delta_{-t})/2$, where δ_t denotes the Dirac measure at t for each $t \in T$. Let λ denote Lebesgue measure on $\mathbb{R}$. We consider two measures on $(T,\mathcal{T})$. The first, μ_1, is λ on T, and the second, μ_2, is λ on T multiplied by the density $2t$. Given any continuous even function ψ on A, and any measure ν on T,

$$\int_T \int_A \psi(a) f(t)(da)\nu(dt) = \int_{[0,1]} \psi(t)\nu(dt).$$

Suppose that there is a g satisfying Equation (22) for $k = 1, 2$. Take $k = 2$ and $\phi(a) = |a|$ on A. Then,

$$\int_0^1 2t\,|g(t)|\,\lambda(dt) = \int_T \int_A |a|\, f(t)(da)\mu_2(dt) = \int_0^1 2t^2 dt = \frac{2}{3}.$$

On the other hand, since $\mu_1 = \lambda$,

$$\int_0^1 (g(t))^2\,\lambda(dt) = \int_T \int_A a^2 f(t)(da)\lambda(dt) = \int_0^1 t^2 dt = \frac{1}{3}.$$

Therefore,

$$\int_0^1 (t - |g(t)|)^2 \lambda(dt) = \int_0^1 t^2 dt - \int_0^1 2t\,|g(t)|\,\lambda(dt) + \int_0^1 (g(t))^2 \lambda(dt) = 0.$$

It follows that $g(t)$ must take the value t or $-t$ λ-a.e. But if g takes the value t on a set $E \in \mathcal{T}$, and θ is the characteristic function of E as a subset of $[-1, 1]$, then

$$\frac{1}{2}\lambda(E) = \int_T \int_A \theta(a) f(t)(da) \mu_1(dt) = \int_T \theta(g(t)) \mu_1(dt) = \lambda(E),$$

so $\lambda(E) = 0$. Similarly $\lambda(T \setminus E) = 0$, and this is impossible.

3. Finite Games with Incomplete Information

As an application of the results in Section 2, we provide a strong purification result for finite games with incomplete information as considered in Milgrom-Weber [**19**]. A *game with incomplete information* Γ consists of a finite set of ℓ players and the following associated spaces and functions. Each player i chooses actions from a compact metric space A_i; the product $\Pi_{j=1}^{\ell} A_j$ is denoted by A. For each player i, a measurable space $(T_i, \mathcal{T}_i)$ represents the personal information and events based on which that player will choose actions from A_i. However, the players' information is incomplete in the sense that they do not know the particulars of the other players' information. The payoff for the i-th player depends on the actions chosen by all the players, and player i's private information $t_i \in T_i$, together with a common state $t_0 \in T_0$ that affects the payoffs of all the players. That is, the i-th player's payoff is given by a function $u_i : A \times T_0 \times T_i \longrightarrow \mathbb{R}$. We assume that T_0 is a finite or countably infinite set $\{t_{0k} : k \in K\}$; $\mathcal{T}_0$ denotes the power set of T_0. The product measurable space $(T, \mathcal{T}) := (\Pi_{j=0}^{\ell} T_j, \Pi_{j=0}^{\ell} \mathcal{T}_j)$ equipped with a probability measure η constitutes the information space of the game Γ. Let η_0 be the marginal probability measure on the countable set T_0, and assume that its only null set is the empty set. We also assume that there is an integrable function α on $(T, \mathcal{T}, \eta)$ such that for each payoff function u_i and each $a \in A$, $u_i(a, t_0, t_i)$ viewed as a function on T is measurable and dominated by α. Note that a boundedness condition on the payoffs is assumed in [**19**, p. 623]. We further assume that each payoff $u_i(\cdot, t_0, t_i)$ is a continuous function on A when t_0 and t_i are fixed.

A *mixed strategy* for player i is a $\mathcal{T}_i$-measurable mapping from T_i to $\mathcal{M}(A_i)$; a *pure strategy* is a $\mathcal{T}_i$-measurable mapping from T_i to A_i. Of course, a pure strategy can also be viewed as a mixed strategy using only Dirac measures. A *mixed (pure) strategy profile* is a collection

$h = \{h_i\}_{i=1}^{\ell}$ of mixed (pure) strategies that specifies a mixed (pure) strategy for each player. In what follows, when i is given, we shall abbreviate a product over all indices $1 \le j \le \ell$ except for $j = i$ by $\Pi_{j \ne i}$; i.e., $\Pi_{j\ne i}$ means $\Pi_{1\le j\le \ell, j\ne i}$. We shall use the following (conventional) notation: $A_{-i} = \Pi_{j\ne i} A_j$, $T_{-i} = \Pi_{j\ne i} T_j$, $a = (a_i, a_{-i})$ for $a \in A$, $t_{-0} = (t_1, \ldots, t_\ell) = (t_i, t_{-i})$ for $(t_0, t_1, \ldots, t_\ell) \in T$, and (h_i, h_{-i}) denotes the strategy profile h.

Assume that the players play the mixed strategy profile $f = \{f_i\}_{i=1}^{\ell}$. Then, the resulting expected payoff for player i is

$$U_i(f) := \int_T \int_A u_i(a, t_i, t_0) f_1(t_1)(da_1) \cdots f_\ell(t_\ell)(da_\ell)\eta(dt), \tag{23}$$

where for each $t \in T$, the inside integral on A is the iterated integral

$$\int_{A_\ell} \cdots \int_{A_1} u_i(a_1, \ldots, a_\ell, t_i, t_0) f_1(t_1)(da_1) \cdots f_\ell(t_\ell)(da_\ell).$$

The mixed strategy profile f is a Nash equilibrium for the game Γ if for each player i, $U_i(f_i, f_{-i}) \ge U_i(f'_i, f_{-i})$ for any other mixed strategy f'_i player i can choose. By introducing an appropriate dummy player 0 with constant payoff u_0 and information space $(T_0, \mathcal{T}_0)$, one can show as a corollary of Theorem 3.1 in [**3**] that there exists a mixed strategy profile that is a Nash equilibrium for the game Γ.

The marginal probability measure of η on $(T_j, \mathcal{T}_j)$ will be denoted by η_j for $0 \le j \le \ell$. For the principal result of this section, we will need a condition on the probability measure η. For each $t_{0k} \in T_0$, $k \in K$, let $\eta(\cdot; t_{0k})$ denote the conditional probability measure on the space $(\Pi_{j=1}^{\ell} T_j, \Pi_{j=1}^{\ell} \mathcal{T}_j)$ when $t_0 = t_{0k}$; such a conditional probability measure always exists since T_0 is countable. For each player i, let $\eta_i(\cdot; t_{0k})$ be the marginal probability measure of $\eta(\cdot; t_{0k})$ on the space $(T_i, \mathcal{T}_i)$. Following [**14**] and [**19**], we shall assume that

$$\eta(\cdot; t_{0k}) = \Pi_{i=1}^{\ell} \eta_i(\cdot; t_{0k}). \tag{24}$$

The latter equality is simply a formulation of the intuitive statement that conditioned on $t_0 \in T_0$, each player's information is independent of the information of the other players. We shall denote the measure $\eta_i(\cdot; t_{0k})$ on $(T_i, \mathcal{T}_i)$ by μ_{ik}.

The following definition has been introduced in [**14**].

Definition 3.1. *A pure strategy profile $g = \{g_i\}_{i=1}^{\ell}$ is said to be a strong purification of the mixed strategy profile $f = \{f_i\}_{i=1}^{\ell}$ if the following four conditions are satisfied for each player i.*

(1) $U_i(f) = U_i(g)$.
(2) *For any given mixed strategy f'_i of player i, $U_i(f'_i, f_{-i}) = U_i(f'_i, g_{-i})$.*

(3) *For each $k \in K$, g_i and f_i have the same conditional distribution on the action space A_i given that $t_0 = t_{0k}$; i.e.,*

$$\int_{T_i} f_i(t_i)(\cdot)\mu_{ik}(dt_i) = \int_{T_i} g_i(t_i)(\cdot)\mu_{ik}(dt_i) = \mu_{ik} g_i^{-1}(\cdot).$$

(4) *for η_i-almost all $t_i \in T_i$, $g_i(t_i) \in \operatorname{supp} f_i(t_i)$.*

Item (2) above says that the expected payoff of player i from the choice of an arbitrary mixed strategy is the same irrespective of whether the opponents play f_{-i} or g_{-i}. It is thus clear that if two strategy profiles satisfy Items (1) and (2) and one is an equilibrium of the game Γ, so is the other.

Now we can apply the results of Section 2 to obtain a strong purification of any mixed strategy profile.

THEOREM 3.2. *Assume that (1) each player i's information is independent of the information of the other players conditioned on the common information as in Equation (24); (2) the marginal probability measure η_i of η on $(T_i, \mathcal{T}_i)$ is a nonatomic Loeb measure. Then every mixed strategy profile f for the game Γ has a strong purification.*

Proof: Fix player i. For each $k \in K$, let λ_k be the positive probability weight $\eta_0(\{t_{0k}\})$. It is clear that for each $S_i \in \mathcal{T}_i$, $\eta_i(S_i) = \sum_{k\in K} \lambda_k \mu_{ik}(S_i)$. Thus, each μ_{ik} is absolutely continuous with respect to η_i; let β_{ik} be the Radon-Nikodym derivative of μ_{ik} with respect to η_i.

Based on our assumption for η of conditional independence as given by Equation (24), the expected payoff $U_i(f)$ of player i for a mixed strategy profile f given by Equation (23) is (recall that t_{-0} represents $(t_1, \ldots, t_\ell)$)

$$\sum_{k\in K} \lambda_k \int_{t_{-0}\in \Pi_{j=1}^{\ell} T_j} \int_{a\in \Pi_{j=1}^{\ell} A_j} u_i(a, t_i, t_{0k}) \Pi_{j=1}^{\ell} f_j(t_j)(da) \Pi_{j=1}^{\ell} \mu_{jk}(dt_{-0}) \tag{25}$$

which means that

$$U_i(f) = \int_{T_i} \int_{A_i} \psi_i^f(t_i, a_i) f_i(t_i)(da_i) \eta_i(dt_i), \tag{26}$$

where $\psi_i^f(t_i, a_i)$ (which depends on the mixed strategy profile f) equals

$$\sum_{k\in K} \lambda_k \beta_{ik}(t_i) \int_{T_{-i}} \int_{A_{-i}} u_i(a_i, a_{-i}, t_i, t_{0k}) \Pi_{j\neq i} f_j(t_j)(da_{-i}) \Pi_{j\neq i} \mu_{jk}(dt_{-i}). \tag{27}$$

For each $j = 1, \cdots, \ell$, denote the measure $\int_{T_j} f_j(t_j, \cdot)\mu_{jk}(dt_j)$ on A_j by $\gamma_{jk}^{f_j}$. Then, from Formula (27) we obtain

$$(28)\quad \psi_i^f(t_i, a_i) = \sum_{k\in K} \lambda_k \beta_{ik}(t_i) \int_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i}, t_i, t_{0k}) d\Pi_{j\neq i}\gamma_{jk}^{f_j}(a_{-i}).$$

Equations (26) and (28) imply that the i-th player's expected payoff depends on the actions of the other players only through the conditional distributions (given $t_0 = t_{0k}$) of their strategies induced on their action spaces.

Recall that α is the η-integrable function that dominates all the payoff functions. Let α_i be the function from T_i to $\mathbb{R}_+$ such that for each $t_i \in T_i$,

$$(29)\qquad \alpha_i(t_i) = \sum_{k\in K} \lambda_k \beta_{ik}(t_i) \int_{t_{-i}\in T_{-i}} \alpha(t_{0k}, t_i, t_{-i}) \Pi_{j\neq i}\mu_{jk}(dt_{-i}).$$

By the Fubini property, it is clear that α_i is η_i-integrable and that $\int_T \alpha(t)\eta(dt) = \int_{T_i} \alpha_i(t_i)\eta_i(dt_i)$. Since, for any $a \in A$, and any $t \in T$, $|u_i(a, t_i, t_0)| \leq \alpha(t_0, t_i, t_{-i})$, Equations (27) and (29) imply that for each $t_i \in T_i$, $a_i \in A_i$, $\psi_i^f(t_i, a_i) \leq \alpha_i(t_i)$. The function $\psi_i^f(\cdot, a_i)$ is obviously measurable on T_i, and the function $\psi_i^f(t_i, \cdot)$ is continuous on A_i.

We now apply Corollary 2.4. The Loeb probability space $(T_i, \mathcal{T}_i, \eta_i)$ and the function ψ_i^f here correspond respectively to the Loeb probability space $(T, \mathcal{T}, P)$ and the functions $\psi_j, j \in J$ in Corollary 2.4. The objects $\mu_{ik}, k \in K$, A_i and f_i correspond to those objects in Corollary 2.4 by dropping the sub-index i. By Corollary 2.4, there exists a pure strategy g_i for player i such that for all $k \in K$,

(i) $\int_{T_i}\int_{A_i} \psi_i^f(t_i, a_i) f_i(t_i)(da_i)\eta_i(dt_i) = \int_{T_i} \psi_i^f(t_i, g_i(t_i))\eta_i(dt_i)$;
(ii) for all Borel set B in A_i,
$\int_{T_i} f_i(t_i)(B)\mu_{ik}(dt_i) = \mu_{ik} g_i^{-1}(B) = \gamma_{ik}^{f_i}$;
(iii) $g_i(t_i) \in \operatorname{supp} f_i(t_i)$ for η_i-almost all $t_i \in T_i$.

Applying the above procedure to each player i, we obtain a pure-strategy profile $g = (g_1, \cdots, g_\ell)$. Now it follows from (ii) and (iii) above that (3) and (4) in Definition 3.1 are satisfied.

To show (1) and (2) in Definition 3.1 are satisfied, consider any mixed strategy f_i' for a fixed player i. Let (f_i', f_{-i}) be denoted by f' and (f_i', g_{-i}) by g'.

By Equation (26), the expected payoffs of player i with f', g and g' are, respectively, given by

$$(30)\qquad U_i(f') = \int_{T_i}\int_{A_i} \psi_i^{f'}(t_i, a_i) f_i'(t_i)(da_i)\eta_i(dt_i),$$

$$U_i(g) = \int_{T_i} \psi_i^g(t_i, g_i(t_i))\eta_i(dt_i), \tag{31}$$

$$U_i(g') = \int_{T_i}\int_{A_i} \psi_i^{g'}(t_i, a_i) f_i'(t_i)(da_i)\eta_i(dt_i). \tag{32}$$

Since Item (ii) above holds for all players, it is obvious that for $j \neq i$, $\gamma_{jk}^{f_j} = \gamma_{jk}^{g_j}$. By Equation (28), ψ_i^f only depends on the probability distributions $\gamma_{jk}^{f_j}, j \neq i$. Hence, we have $\psi_i^f = \psi_i^g = \psi_i^{f'} = \psi_i^{g'}$. By Item (i) above together with Equations (26) and (31), it follows that

$$\begin{aligned} U_i(f) &= \int_{T_i}\int_{A_i} \psi_i^f(t_i, a_i) f_i(t_i)(da_i)\eta_i(dt_i) = \int_{T_i} \psi_i^f(t_i, g_i(t_i))\eta_i(dt_i) \\ &= \int_{T_i} \psi_i^g(t_i, g_i(t_i))\eta_i(dt_i) = U_i(g). \end{aligned}$$

This means that (1) in Definition 3.1 holds. Similarly,

$$\begin{aligned} U_i(f') &= \int_{T_i}\int_{A_i} \psi_i^{f'}(t_i, a_i) f_i'(t_i)(da_i)\eta_i(dt_i) \\ &= \int_{T_i}\int_{A_i} \psi_i^{g'}(t_i, a_i) f_i'(t_i)(da_i)\eta_i(dt_i) = U_i(g'), \end{aligned}$$

whence, (2) in Definition 3.1 holds, and we are done. □

Remark 3.3. In Section 4 of [**14**], a finite-player game Γ_0 with finite action spaces, and diffuse and mutually independent private information, as formulated by Radner-Rosenthal [**20**], is reformulated as a special case of the finite-action game considered in [**19**]. That allows a synthetic treatment of finite-player and finite-action games with private information that is independent or conditionally independent. It is shown in Theorem 2 of [**14**] that in the game Γ_0 with finite action spaces, every mixed strategy profile has a strong purification. When the game Γ_0 has compact metric action spaces, Theorem 3 in [**15**] shows the existence of pure-strategy equilibria for Γ_0 in the case of nonatomic Loeb information spaces. By viewing a game Γ_0 with compact metric action spaces as a special case of the game Γ considered in this section, Theorem 3.2 then implies that under the conditions of Theorem 3 in [**15**], every mixed strategy profile in the game Γ_0 with compact metric action spaces has a strong purification.

Remark 3.4. As noted in the third paragraph of this section, a Nash equilibrium in mixed strategies exists in the game Γ. Theorem 3.2 then implies that a Nash equilibrium in pure strategies exists in the game Γ with nonatomic Loeb probability spaces modeling information.

The example in [**13**] presents a two-player game with the Lebesgue unit square as the joint information space and the interval $[-1,1]$ as the action space for both players; it has no Nash equilibrium. Thus, for games Γ and Γ_0 with compact metric action spaces, both the strong purification result and the existence of a pure-strategy Nash equilibrium can fail if we remove the restriction that the private information spaces are nonatomic Loeb probability spaces. A general purification result is claimed by Fudenberg and Tirole in [**11**, Theorem 6.2, p. 236]. This result holds when the private information spaces are nonatomic Loeb probability spaces as shown in Theorem 3.2; it fails otherwise as in [**13**].

References

[1] S. Albeverio, J. E. Fenstad, R. Høegh Krohn, T. Lindstrøm, *Nonstandard methods in stochastic analysis and mathematical physics*, Academic Press, Orlando, 1986.

[2] C. D. Aliprantis and K. C. Border, Infinite dimensional analysis: a hitchhiker's guide, Springer-Verlag, Berlin, 1994.

[3] E. J. Balder, Generalized equilibrium results for games with incomplete information, *Math. Oper. Res.* **13** (1988), 265–276.

[4] M. Balhorn, The rise of epicene *they*, *Journal of English Linguistics*, **32**(2004), 79–104. http://eng.sagepub.com/cgi/content/refs/32/2/79

[5] V. Bergström, Ein neuer Beweis eines Satzes von E. Steinitz, *Abh. Math. Sem. Univ. Hamburg.* **8** (1931), 148–154.

[6] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[7] N. J. Cutland, Internal controls and relaxed controls, *J. London Math. Soc.* **27** (1983), 130–140.

[8] A. Dvoretsky, A. Wald and J. Wolfowitz, Elimination of randomization in certain problems of statistics and of the theory of games, *Proc. Nat. Acad. Sci. USA* **36** (1950), 256–260.

[9] A. Dvoretsky, A. Wald and J. Wolfowitz, Relations among certain ranges of vector measures, *Pac. J. Math.* **1** (1951), 59–74.

[10] A. Dvoretsky, A. Wald and J. Wolfowitz, Elimination of randomization in certain statistical decision problems in certain statistical decision procedures and zero-sum two-person games, *Ann. Math. Stat.* **22** (1951), 1–21.

[11] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, Cambridge, 1991.

[12] H. J. Keisler, An infinitesimal approach to stochastic analysis, *Memoirs Amer. Math. Soc.* **48** (1984).

[13] M. A. Khan, K. P. Rath and Y. N. Sun, On a private information game without pure strategy equilibria, *J. Math. Econ.* **31** (1999), 341–359.

[14] M. A. Khan, K. P. Rath and Y. N. Sun, The Dvoretzky-Wald-Wolfowitz theorem and purification in atomless finite-action games, submitted.

[15] M. A. Khan and Y. N. Sun, Non-cooperative games on hyperfinite Loeb spaces, *J. Math. Econ.* **31** (1999), 455–492.

[16] P. A. Loeb, A combinatorial analog of Lyapunov's Theorem for infinitesimally generated atomic vector measures, *Proc. Amer. Math. Soc.* **39** (1973) 585–586.

[17] P. A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, *Trans. Amer. Math. Soc.* **211** (1975), 113–122.

[18] P. A. Loeb and M. Wolff, ed., *Nonstandard Analysis for the Working Mathematician*, Kluwer Academic Publishers, Amsterdam, 2000.

[19] Milgrom, P. R., and R. J. Weber, Distributional strategies for games with incomplete information, *Math. Oper. Res.* **10** (1985), 619–632.

[20] R. Radner and R. W. Rosenthal, Private information and pure-strategy equilibria, *Math. Oper. Res.* **7** (1982), 401–409.

[21] D. A. Ross, An elementary proof of Lyapunov's theorem, to appear in the *American Mathematical Monthly*.

[22] H. L. Royden, *Real Analysis*, third edition, Macmillan, New York, 1988.

[23] Y. N. Sun, Distributional properties of correspondences on Loeb spaces, *J. Funct. Anal.*, **139** (1996), 68–93.

[24] E. Steinitz, Bedingt convergente Reihen und konvexe Systeme, *J. Reine Angew. Math.* **143** (1913), 128–175.

Department of Mathematics, University of Illinois, Uabana, IL 61801. e-mail: loeb@math.uiuc.edu

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543; and Department of Economics, National University of Singapore, 1 Arts Link, Singapore 117570. e-mail: matsuny@nus.edu.sg