A LOCAL MAXIMAL FUNCTION SIMPLIFYING MEASURE DIFFERENTIATION

JÜRGEN BLIEDTNER AND PETER A. LOEB

1. Introduction.

Differentiation and absolute continuity are difficult topics for students in a graduate integration course. We outline here a simplified approach that has cut almost in half the time previously spent by the second author in covering these topics (as developed in chapter 5 of [7]); the time saved is used for a deeper discussion of Hilbert spaces and Fourier analysis. Our simplified approach uses a local maximal function. It was originally developed and employed by the authors in [2], [3], and [4] to deal with limit theorems in various settings. The advantage gained is that many limit results can be established just by proving them for sets where the relevant input vanishes.

2. Measures from Integrators.

We work with Lebesgue measure λ on a bounded open interval $J = (-N, N)$. In the development of the Lebesgue integral, most proofs establish at the same time equivalent results for a general Borel measure μ_F and the μ_F-integral. Here, μ_F is formed from an increasing, right-continuous integrator F by replacing the length of each interval $(a, b]$ with the change of F. We assume that $\lim_{x\to(-N)+} F(x)$ and $\lim_{x\to N-} F(x)$ are finite. If we start with a finite Borel measure ν on J, then $F_\nu(x) = \nu(-N, x]$ defines an integrator with $\nu = \mu_{F_\nu}$.

3. Lusin's Theorem.

We need Lusin's theorem. Here is a proof by Erik Talvila and the second author from [5], restricted to the case of Lebesgue measure and the real line $\mathbb{R}$. It is simpler than one based on Egoroff's theorem.

Theorem 3.1 (**Lusin**). *Let f be a Lebesgue measurable, real-valued function on J. For each $\varepsilon > 0$ there is a compact subset K of J with $\lambda(J \setminus K) < \varepsilon$ such that f restricted to K is continuous.*

Date: From June-July, 2007 MAA Monthly.

Proof. Let $\langle V_n : n \in \mathbb{N}\rangle$ be an enumeration of the open intervals in $\mathbb{R}$ with rational endpoints. Fix compact subsets K_n of $f^{-1}[V_n]$ and K'_n of $J \setminus f^{-1}[V_n]$ for each n so that $\lambda(J \setminus K) < \varepsilon$ when $K := \bigcap_n (K_n \cup K'_n)$. Given x in K and an n with $f(x)$ in V_n, we see that x belongs to the open set $O := J \setminus K'_n$ and $f[O \cap K]$ is contained in V_n. □

4. The optimal covering theorem for $\mathbb{R}$.

We also use the following optimal covering theorem for the real line developed by J. Aldaz [1] from a lemma of T. Radó [6]. In [1], the constant 3 is improved to $2+\varepsilon$ for an arbitrary $\varepsilon > 0$.

Theorem 4.1 (Radó-Aldaz). *Let μ be a finite Borel measure on J. If $\mathcal{I}$ is an arbitrary collection of nondegenerate intervals, all contained in J, then the set $\cup_{I\in\mathcal{I}} I$ is measurable, and there is a finite disjoint subset $\{I_1, \cdots, I_j\}$ of $\mathcal{I}$ such that*

$$\mu(\cup_{I\in\mathcal{I}} I) \leq 3 \cdot \Sigma_{i=1}^{j} \mu(I_i).$$

Proof. It is easy to see that $(\cup_{I\in\mathcal{I}} I) \setminus (\cup_{I\in\mathcal{I}} I^\circ)$ is at most a countable set. By Lindelöf's theorem we may assume that $\mathcal{I}$ itself is a countable, ordered collection $\{I_n\}$ with measurable union having finite measure. Choose m in $\mathbb{N}$ so that

$$\frac{3}{2} \cdot \mu(\cup_{n=1}^{m} I_n) \geq \mu(\cup_{n=1}^{\infty} I_n) = \mu(\cup_{I\in\mathcal{I}} I).$$

We now employ the method of Radó's lemma by discarding redundant intervals so that each remaining interval I_n with $n \leq m$ contains a point x not in any other remaining interval I_k with $k \leq m$. We order these points x_i and reorder the corresponding intervals in the same order so that for any indices i, j, and k with $i < j < k$ we have $x_i < x_j < x_k$, and thus $I_i \subseteq (-\infty, x_j)$ and $I_k \subseteq (x_j, +\infty)$. Since the intervals with even indices form a disjoint collection, as do the intervals with odd indices, the desired subset of $\mathcal{I}$ is whichever of these two families has the greater total measure. □

5. A local maximal function.

The principal device of our approach to differentiation is a local maximal function defined in terms of a finite Borel measure μ and Lebesgue measure λ on $J = (-N, N)$. For example, μ may be the measure λ_f generated by λ and a nonnegative integrable function f on J, where $\lambda_f(A) = \int_A f \, d\lambda$ for each Borel subset A of J.

We let $\mathcal{I}(x, r)$ denote to the collection of all nondegenerate intervals I containing x and contained in J with I having length $\lambda(I) \leq r$, and we set

$$M(\mu, r, x) = \sup_{I \in \mathcal{I}(x,r)} \frac{\mu(I)}{\lambda(I)}.$$

Since $M(\mu, r, x)$ decreases as r decreases, we can set

$$M(\mu, x) = \lim_{r \to 0+} M(\mu, r, x).$$

We call $M(\mu, \cdot)$ the *local maximal function* generated by μ and λ.

Recall that the classical maximal function at x for a measure μ takes the value $\sup_{r>0} M(\mu, r, x)$ (see, for example, [8]). As is the case with the classical maximal function, we have the following inequality:

Proposition 5.1. *Let E be a subset of J. For each $\alpha > 0$, let $E_\alpha = \{x \in E : M(\mu, x) > \alpha\}$. Then the Lebesgue outer measure $\lambda^*(E_\alpha)$ satisfies*

$$\lambda^*(E_\alpha) \leq \frac{3}{\alpha} \cdot \mu(J).$$

Proof. Each point x in E_α is contained in an interval I_x in $\mathcal{I}(x, 1)$ such that $\alpha \cdot \lambda(I_x) \leq \mu(I_x)$. By Theorem 4.1 there is a finite disjoint subcollection $\{I_1, \cdots, I_n\}$ of these intervals such that

$$\lambda^*(E_\alpha) \leq \lambda(\cup_{x \in E_\alpha} I_x) \leq 3 \cdot \Sigma_{k=1}^n \lambda(I_k) \leq \frac{3}{\alpha} \Sigma_{k=1}^n \mu(I_k) \leq \frac{3}{\alpha} \cdot \mu(J). \quad \square$$

Next, we capsulize the advantage of localizing our maximal function.

Theorem 5.2. *If E is a Borel subset of J with $\mu(E) = 0$, then $M(\mu, x) = 0$ for Lebesgue almost all x in E.*

Proof. Fix $\alpha > 0$ and an $\varepsilon > 0$. Since $\mu(E) = 0$, there is an open set U with $E \subseteq U \subseteq J$ such that $\mu(U) < \varepsilon\alpha/3$. Let ν be the finite measure on J defined by setting $\nu(A) = \mu(A \cap U)$ for each Borel subset A of J. For each x in E, $M(\mu, x) = \lim_{r \to 0+} M(\mu, r, x)$ can be calculated with values of r small enough so that all intervals in $\mathcal{I}(x, r)$ lie inside U. For these points x it follows that $M(\mu, x) = M(\nu, x)$, whence

$$E_\alpha = \{x \in E : M(\mu, x) > \alpha\} = \{x \in E : M(\nu, x) > \alpha\}.$$

Therefore,

$$\lambda^*(E_\alpha) \leq \frac{3}{\alpha} \nu(J) = \frac{3}{\alpha} \mu(U) < \varepsilon.$$

Since ε is arbitrary, $\lambda^*(E_\alpha) = 0$. Since α is arbitrary, the result follows. $\square$

Corollary 5.3. *Let λ_f be the finite measure on J generated by Lebesgue measure λ and a nonnegative integrable function f. If $f(x) = 0$ for Lebesgue almost all points of a Borel subset E of J, then $M(\lambda_f, x) = 0$ for Lebesgue almost all x in E.*

6. Differentiation.

Our next, and principal, result shows the need to consider coverings for which each point of the set that is covered may be an endpoint of the corresponding covering interval.

Theorem 6.1 (Lebesgue Differentiation Theorem). *For every Lebesgue integrable function f on $\mathbb{R}$, each of the following equalities holds Lebesgue almost everywhere on $\mathbb{R}$:*

$$\begin{aligned}\lim_{r\to 0+} \frac{1}{2r} \int_{[x-r,x+r]} f\, d\lambda &= f(x),\\ \lim_{r\to 0+} \frac{1}{r} \int_{[x,x+r]} f\, d\lambda &= f(x),\\ \lim_{r\to 0+} \frac{1}{r} \int_{[x-r,x]} f\, d\lambda &= f(x).\end{aligned}$$

Proof. We need only prove the result when $f \geq 0$ and for points of $J = (-N, N)$. By Lusin's theorem, for each $\varepsilon > 0$, there is a compact subset K of J with $\lambda(J \setminus K) < \varepsilon$ such that $f|K$ is continuous on K. We can extend $f|K$ with a nonnegative, bounded function g that vanishes on K so that $h := f \cdot \chi_K + g$ is continuous on J. Now on J,

$$f = h - g + f \cdot \chi_{J\setminus K}.$$

By Corollary 5.3 and the continuity of h, each limit result holds almost everywhere on K for h, g, and $f \cdot \chi_{J\setminus K}$, and thus for f. Since ε is arbitrary, the limit results are established for almost all points of J. □

Corollary 6.2. *If f is λ-integrable on $[a, b]$ and $F : [a, b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f\, d\lambda + C$ where C is a constant, then $F'(x) = f(x)$ for λ-almost all x in $[a, b]$.*

To differentiate general integrators, we need the following result for a finite measure ν on $J = (-N, N)$ and integrator F_ν, where $F_\nu(x) = \nu((-N, x])$ at each x in J.

Theorem 6.3. *If ν is a finite Borel measure on J and E is a Borel subset of J with $\nu(E) = 0$, then F_ν has a zero derivative Lebesgue almost everywhere on E.*

Proof. Since $\nu(E) = 0$, $\nu(\{a\}) = 0$ for each point a of E. Moreover, by Theorem 5.2, $M(\nu, a) = 0$ for λ-almost all points a in E. Given such a point a and small $\Delta x > 0$, we have

$$\begin{aligned}
\frac{F_\nu(a + \Delta x) - F_\nu(a)}{\Delta x} &= \frac{\nu(a, a + \Delta x]}{\Delta x} = \frac{\nu[a, a + \Delta x]}{\Delta x}, \\
\frac{F_\nu(a - \Delta x) - F_\nu(a)}{-\Delta x} &= \frac{F_\nu(a) - F_\nu(a - \Delta x)}{\Delta x} = \frac{\nu(a - \Delta x, a]}{\Delta x}.
\end{aligned}$$

Since both ratios have limit 0 as $\Delta x \to 0$, $F'_\nu(a)$ exists and is 0. □

7. ABSOLUTE CONTINUITY.

In the development of absolute continuity, we apply the Radon-Nikodym derivative theorem and its corollary, the Lebesgue decomposition theorem, but the proofs are postponed until the assignment of exercises on Hilbert spaces (see [7, p. 280]). The treatment of functions of bounded variation is the usual one, except almost everywhere differentiability is established somewhat later. Similarly, the introduction of absolutely continuous functions is standard.

That introduction leads to the relationship between absolute continuity of functions and the absolute continuity of measures with respect to Lebesgue measure. Because we are working with finite measures on a bounded interval $J = (-N, N)$, a Borel measure μ is absolutely continuous with respect to λ (in symbols $\mu << \lambda$) if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\lambda E < \delta$ implies that $\mu E < \varepsilon$. Moreover, for the Borel measure μ_F obtained from an integrator F on J, $\mu_F << \lambda$ if and only if F is absolutely continuous. In this case, it follows from the Radon-Nikodym derivative theorem that there is a nonnegative Lebesgue integrable function f on J such that $F(x) = \int_{(-N,x]} f \, d\lambda$ for each x in J, whence by Theorem 6.1 $F'(x)$ exists and equals $f(x)$ for almost all x in J.

It is easy to show that an absolutely continuous function f on an interval $[a, b]$ is of bounded variation, and its total variation $T_a^x f$, positive variation $P_a^x f$, and negative variation $N_a^x f$ are all absolutely continuous on $[a, b]$. It follows that any such f is the difference of two nonnegative, increasing (by this we mean nondecreasing) absolutely continuous functions on $[a, b]$. Moreover, we can always extend f so that it is constant above b and below a, and thus absolutely continuous on J. We therefore have the following result:

Theorem 7.1. *If F is absolutely continuous on $[a, b]$, then its derivative F' exists Lebesgue almost everywhere on $[a, b]$, and*

$$F(x) = F(a) + \int_a^x F' \, d\lambda$$

for λ-almost all x in $[a, b]$.

Finally, to show that any increasing real-valued function (hence any function of bounded variation) has a derivative Lebesgue almost everywhere, we consider an increasing function $F : J \to \mathbb{R}$. We assume that F has finite limits at the endpoints $-N$ and N of J. It is an interesting exercise to show that if G is an increasing function equal to F except at points where F jumps, then $F' = G'$ at points where either derivative exists. We may assume, therefore, that F is a right-continuous integrator generating a finite measure. By the Lebesgue decomposition theorem that measure is the sum of a measure μ such that $\mu << \lambda$ and a measure ν for which there is a Lebesgue null set A contained in J with $\nu(J \setminus A) = 0$. It follows from Theorem 6.3 that the cumulative distribution function F_ν has a zero derivative Lebesgue almost everywhere in J. Since F_μ has a derivative Lebesgue almost everywhere in J, the same is true for $F = F_\mu + F_\nu$. Thus, with the inequality proved as in [7, p. 101], we have the following result:

Theorem 7.2. *An increasing real-valued function F on an interval $[a, b]$ has a derivative f Lebesgue almost everywhere on $[a, b]$, and $\int_a^b f \, d\lambda \leq F(b) - F(a)$. The inequality can be replaced with equality if and only if F is absolutely continuous on $[a, b]$.*

References

[1] J. M. Aldaz, A general covering lemma for the real line, *Real Anal. Exchange* **17** (1991/92) 394–398.

[2] J. Bliedtner and P. A. Loeb, A reduction technique for limit theorems in analysis and probability theory, *Arkiv för Mat.* **30** (1992) 25–43.

[3] _________, Best filters for the general Fatou boundary limit theorem, *Proc. Amer. Math. Soc.* **123** (1995) 459–463.

[4] _________, The optimal differentiation basis and liftings of L^∞, *Trans. Amer. Math. Soc.* **352** (2000) 4693–4710.

[5] P. A. Loeb and E. Talvila, Lusin's theorem and Bochner integration, *Sci. Math. Jpn.* **60** (2004) 113–120.

[6] T. Radó, Sur un problème relatif à un théorème de Vitali, *Fund. Math.* **11** (1928) 228–229.

[7] H. L. Royden, *Real Analysis*, 3rd ed., Macmillan, New York, 1988.

[8] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

Fachbereich Informatik und Mathematik, Universität Frankfurt, Robert-Mayer-Str. 6-8, D-60054, Frankfurt/M, FRG
E-mail address: bliedtne@math.uni-frankfurt.de

Department of Mathematics, University of Illinois, 1409 West Green St., Urbana, Ill. 61801, USA
E-mail address: loeb@math.uiuc.edu