A General Fatou Lemma

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Abstract. A general Fatou Lemma is established for a sequence of Gelfand integrable functions from a vector Loeb space to the dual of a separable Banach space or, with a weaker assumption on the sequence, a Banach lattice. A corollary sharpens previous results in the finite dimensional setting even for the case of scalar measures. Counterexamples are presented to show that the results obtained here are sharp in various aspects. Applications include systematic generalizations of the distribution of correspondences from the case of scalar Loeb spaces to the case of vector Loeb spaces and a proof of the existence of a pure strategy equilibrium in games with private and public information and with compact metric action spaces.

1. Introduction

Fatou's Lemma for a sequence of real-valued integrable functions is a basic result in real analysis. Its finite-dimensional generalizations have also received considerable attention in the literature of mathematics and economics; see, for example, [**12**], [**13**], [**20**], [**26**], [**28**] and [**31**].

A main aim of this paper is to establish a general Fatou type result for a sequence of Gelfand integrable functions from a vector Loeb measure space to the dual of a separable Banach lattice; such a result fails when the Loeb measure space (first developed in [**21**]) is replaced by other spaces such as a Lebesgue measure space. In particular, we

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consider a sequence $\{g_n\}_{n=1}^{\infty}$ of Gelfand integrable, non-negative functions mapping a measurable space $(\Omega, \mathcal{A})$ supplied with countably many Loeb measures μ_j, $j \in J$, into a dual Banach lattice X. An additional weak*-tightness condition on the sequence $\{g_n\}_{n=1}^{\infty}$ can be dropped if X satisfies a "norm approximation property". Forming a control measure $\bar{\mu}$ from the measures μ_j, we show that if the Gelfand integrals $\int_\Omega g_n d\mu_j$ have a weak* limit $a_j \in X$ as n goes to infinity, then there is a function g from $(\Omega, \mathcal{A})$ to X such that for $\bar{\mu}$-a.e. $\omega \in \Omega$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^{\infty}$, and g is Gelfand μ_j-integrable with $\int_\Omega g d\mu_j \le a_j$ for each $j \in J$. If in addition the sequence $\{g_n\}_{n=1}^{\infty}$ satisfies a strong condition, that of uniform integrability, then one can claim $\int_\Omega g d\mu_j = a_j$ for each $j \in J$ without imposing a lattice structure on X. These results will be stated in Theorem 2.4 below, part of which will be generalized in Theorem 2.5 to a setting where the sequence $\{g_n\}_{n=1}^{\infty}$ has a uniformly integrable sequence of lower bounds.

One motivation of these results comes from game theory. The fundamental existence result of John Nash [**25**] is only shown for mixed strategy equilibria in a game with finitely many players, where players choose probability distributions over their actions. The proof of Nash's theorem is based on Kakutani's Fixed Point Theorem for correspondences [**15**]. A problem of great interest in game theory is finding conditions that will guarantee the existence of equilibria in pure strategies (i.e., strategies taking values in the action spaces themselves) for games with incomplete information. The example in [**17**], however, constructs a simple game between two players with its independent and diffuse information modeled by the product of two Lebesgue unit intervals, but with no pure strategy equilibrium. In order to show the existence of a pure strategy equilibrium in Theorem 4.1 for a general finite-player game with both public and private information, where the players' private information spaces are modeled by Loeb spaces, we also develop systematic generalizations of the distribution of correspondences from the case of scalar Loeb spaces developed in [**29**] to the case of vector Loeb spaces; for this we apply Theorem 2.4 to the special case of a space of measures. The proof of Theorem 4.1 here uses the infinite dimensional Fixed Point Theorem of Fan [**9**] and Glicksberg [**11**] for correspondences together with the properties of compactness, convexity and preservation of upper semicontinuity of distribution of correspondences shown in Section 3.

The paper is organized as follows. Section 2 presents the general results of Fatou type, their simple consequences, and several counterexamples showing that our Fatou type results are sharp in various aspects. The distribution theory of correspondences on vector Loeb spaces is developed in Section 3. We show the existence of a pure strategy equilibrium in games with private and public information and with compact metric action spaces in Section 4. The proofs of Theorems 2.4 and 2.5, which need some subtle arguments from nonstandard analysis, are presented respectively in Sections 5 and 6. We refer the reader to [**23**] for the basics of nonstandard analysis.

2. An Infinite Dimensional Fatou's Lemma

Let Ω be a non-empty internal set, $\mathcal{A}_0$ an internal algebra on Ω, and $\mathcal{A}$ the σ-algebra generated by $\mathcal{A}_0$. Let J be a finite or countably infinite set, μ_{0j}, $j \in J$, internal finitely additive probability measures on $(\Omega, \mathcal{A}_0)$, and μ_j, $j \in J$, the corresponding Loeb probability measures on $(\Omega, \mathcal{A})$.

Let $\bar{\mu}$ be a Loeb probability measure on $(\Omega, \mathcal{A})$ so that for each $j \in J$, μ_j is absolutely continuous with respect to $\bar{\mu}$. For example, when J is finite with cardinality $|J|$, one can set $\bar{\mu}_0(A) = \sum_{j\in J} \frac{1}{|J|}\mu_{0j}(A)$ for each $A \in \mathcal{A}_0$. The Loeb measure of $\bar{\mu}_0$ is simply $\bar{\mu} = \sum_{j\in J} \frac{1}{|J|}\mu_j$. When J is countably infinite, it can be viewed as the set $\mathbb{N}$ of natural numbers. Extending μ_{0j}, $j \in J$, to an internal set μ_{0j}, $j \in {}^*J$, of internal probability measures on $(\Omega, \mathcal{A}_0)$, one sets $\bar{\mu}_0(A) = \sum_{j\in {}^*J} \frac{1}{2^j}\mu_{0j}(A)$ for each $A \in \mathcal{A}_0$. In this case, the Loeb measure of $\bar{\mu}_0$ is simply $\bar{\mu} = \sum_{j\in J} \frac{1}{2^j}\mu_j$. When necessary, we will assume that $\mathcal{A}$ is complete with respect to $\bar{\mu}$.

We shall use the notation $\mathbf{1}_A$ for characteristic function of a set A. The following defines the concept of uniform integrability and the concept of tightness for a sequence of functions with respect to the norm and weak* topologies of a Banach space.

Definition 2.1. *Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions from a probability space $(\Omega, \mathcal{A}, P)$ to a Banach space X.*

(1) The sequence $\{g_n\}_{n=1}^{\infty}$ is said to be uniformly P-norm-integrable if for each $n \in \mathbb{N}$, $\|g_n\|$ is integrable on $(\Omega, \mathcal{A}, P)$, and

$$\lim_{k\to\infty} \sup_n \int_{\|g_n\|\geq k} \|g_n\| dP = 0.$$

(2) When X is the norm dual of a Banach space Y, the sequence $\{g_n\}_{n=1}^{\infty}$ is said to be weak P-tight, if for any $\varepsilon > 0$, there exists a weak* compact set K in X such that for all $n \in \mathbb{N}$, $g_n^{-1}[K] \in \mathcal{A}$ and $P(g_n^{-1}[K]) > 1 - \varepsilon$.*

Definition 2.2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space.*
(1) Let X be the norm dual of a Banach space Y. For $x \in X$, $y \in Y$,

the value of the linear functional x at y will be denoted by $\langle x, y\rangle$. A function f from $(\Omega, \mathcal{A}, P)$ to X is said to be Gelfand P-integrable if for each $y \in Y$, the real-valued function $\langle f(\cdot), y\rangle$ is integrable on $(\Omega, \mathcal{A}, P)$. It follows from the Closed Graph Theorem that there is a unique element $x \in X$ such that $\langle x, y\rangle = \int_\Omega \langle f(\omega), y\rangle \; dP$ for all $y \in Y$ (see [**8**, p. 53]*); that element x, called the Gelfand integral, will be denoted by $\int_\Omega f \; dP$.*

(2) Let X be the dual of a Banach lattice Y. There is a natural dual order (denoted by $\leq$) and lattice norm $|\cdot|$ satisfying the condition $|x| \leq |z| \Rightarrow \|x\| \leq \|z\|$. We will say that X has the ***norm approximation property*** *if there is an increasing (perhaps constant) sequence y_m of nonnegative elements in Y with $\lim_{m\to\infty} \langle x, y_m\rangle = \|x\|$ for each nonnegative $x \in X$.*

Example 2.3. The space ℓ^1 is the dual space of the space c_0 consisting of the continuous functions that vanish at ∞ on the one-point compactification of $\mathbb{N}$. This space has the norm approximation property since for each $n \in \mathbb{N}$, we may set $h_n \in c_0$ equal to the sequence taking the value 1 from 1 to n and 0 thereafter. The space $\mathcal{M}(K)$ consisting of finite, signed Borel measures on a second-countable, locally compact Hausdorff space K also has this property. If K is actually compact, the space $\mathcal{M}(K)$ is the dual space of the space of continuous real-valued functions on K, and for each nonnegative $\mu \in \mathcal{M}(K)$, $\|\mu\| = \langle \mu, 1\rangle$. A sequence similar to that used for ℓ^1 exists if K is just locally compact. A sufficient condition for X to have the norm approximation property will appear in an article [**24**] by Heinrich Lotz.

The following, our principal result, is an exact Fatou Lemma for the vector Loeb measure formed from the Loeb probability measures μ_j on $(\Omega, \mathcal{A})$. It is couched in the general settings of a dual Banach space and dual Banach lattice.

Theorem 2.4. *Let X be the norm dual of a separable Banach space Y, and let $\{g_n\}_{n=1}^\infty$ be a sequence of functions from $(\Omega, \mathcal{A})$ to X. Suppose that for each $j \in J$, each g_n is Gelfand integrable on $(\Omega, \mathcal{A}, \mu_j)$, and as n goes to infinity, the Gelfand integrals $\int_\Omega g_n d\mu_j$ have a weak* limit $a_j \in X$. Then there is a function g from $(\Omega, \mathcal{A})$ to X such that for $\bar{\mu}$-a.e. $\omega \in \Omega$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^\infty$, and g has the following properties for each of the following two cases:*

A. *Suppose that for every $j \in J$, the sequence $\{g_n\}_{n=1}^\infty$ is uniformly μ_j-norm-integrable. It then follows that for each $j \in J$, g is Gelfand μ_j-integrable and $\int_\Omega g d\mu_j = a_j$.*

B. *Suppose that Y is in fact a separable Banach lattice, and X is its dual Banach space with the natural dual order and lattice norm and for* *each $n \in \mathbb{N}$ and $\omega \in \Omega$, $g_n(\omega) \geq 0$. We assume either that X has the norm approximation property, or if that assumption does not hold, that the sequence $\{g_n\}_{n=1}^{\infty}$ is weak* μ_j-tight. It then follows that:*

1. g is Gelfand μ_j-integrable with $\int_\Omega g d\mu_j \leq a_j$ for each $j \in J$;

2. $\int_\Omega \langle g, y\rangle d\mu_j = \langle a_j, y\rangle$ for any $y \in Y$ and $j \in J$ for which $\{\langle g_n, y\rangle\}_{n=1}^{\infty}$ is uniformly μ_j-integrable;

3. in particular, $\int_\Omega g\ d\mu_j = a_j$ for any $j \in J$ for which $\{g_n\}_{n=1}^{\infty}$ is uniformly μ_j-norm-integrable.

Case A of Theorem 2.4 generalizes a version of Theorem 10 in [**30**] for a sequence of Gelfand integrable functions for which the norms are dominated by an integrable function on a scalar Loeb measure space to the case for which the functions are uniformly norm-integrable on a vector Loeb measure space. Case B of Theorem 2.4 can be generalized with the following result. We present the statement and proof of the generalization as a separate result so that Theorem 2.4 and its proof can be given in a simpler form.

THEOREM 2.5. *The conclusions of Case B of Theorem 2.4 still hold if the assumption that for each $n \in \mathbb{N}$ and $\omega \in \Omega$, $g_n(\omega) \geq 0$ is replaced with the more general assumption that there is a sequence $\{f_n\}_{n=1}^{\infty}$ of functions from $(\Omega, \mathcal{A})$ to X with the following properties:*

1. For each $n \in \mathbb{N}$ and $\omega \in \Omega$, $g_n(\omega) \geq f_n(\omega)$.

2. For each $n \in \mathbb{N}$ and $j \in J$, f_n is Gelfand μ_j-integrable.

3. The sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly μ_j-norm-integrable for each $j \in J$.

REMARK 2.6. (1) When a general measure space is used instead of a Loeb space, Examples 2.9 and 2.10 below together with Example 2 in [**30**] show that both Cases A and B in Theorem 2.4 may fail to hold. The previous literature has considered only approximate versions of the conclusion in Case A of Theorem 2.4 in an infinite dimensional setting for the case of a general scalar measure; see, for example, [**4**], [**5**], [**16**], [**32**] and [**33**]. Moreover, neither approximate nor exact versions of our results in Case B of Theorem 2.4 and in Theorem 2.5 in the infinite dimensional setting have been considered before even for a scalar measure.

(2) On the other hand, if one does work with general probability spaces, one can consider their nonstandard extensions and the corresponding Loeb spaces. By transferring the exact Fatou type result in

Theorem 2.5, one can obtain an approximate version of the Fatou type result in Theorem 2.5 for general probability spaces.

(3) Theorems 2.4 and 2.5 continue to hold if the Loeb probability measures are replaced by finite positive Loeb measures.

(4) By using the saturation property from [**14**], the authors plan to extend the results of this paper to general nonatomic probability spaces that are nowhere countably generated in the sense that the σ-algebra of measurable sets in any set of positive measure can not be countably generated.

The next result for the case that $X = \mathbb{R}^p$ generalizes earlier results on Fatou's Lemma for a scalar measure (see, for example, [**3**] and [**28**]) to the case of vector measures with a finite or countably infinite number of component scalar measures. Even for the case of one scalar measure, it sharpens previous results by showing that equality can be achieved at any particular component of an $\mathbb{R}^p$-valued function satisfying a uniform integrability condition even when not all components satisfy such a condition. For the norm of each $x = (x^1, \ldots, x^p) \in \mathbb{R}^p$ we take the value $\Sigma_{i=1}^p |x^i|$. The space $\mathbb{R}^p$ has the norm approximation property since when x has nonnegative components,

$$\|x\| = \Sigma_{i=1}^p x^i \cdot 1 = \langle x, 1 \rangle .$$

COROLLARY 2.7. *Let $\{g_n\}_{n=1}^\infty$ be a sequence of functions from $(\Omega, \mathcal{A})$ to $\mathbb{R}^p$ with component functions $\{g_n^i\}_{n=1}^\infty$, $i = 1, \ldots, p$, that are μ_j-integrable for each $j \in J$. Assume that for each $j \in J$, $\lim_{n\to\infty} \int_\Omega g_n d\mu_j = a_j = (a_j^1, \ldots, a_j^p)$. Also assume that there is a sequence $\{f_n\}_{n=1}^\infty$ of measurable functions from $(\Omega, \mathcal{A})$ to $\mathbb{R}^p$ with component functions $\{f_n^i\}_{n=1}^\infty$, $i = 1, \ldots, p$, such that $f_n(\omega) \le g_n(\omega)$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, and moreover, for each $i \le p$ and $j \in J$, $\{f_n^i\}_{n=1}^\infty$ is uniformly μ_j-norm integrable. Then there is a function $g = (g^1, \ldots, g^p)$ from $(\Omega, \mathcal{A})$ to $\mathbb{R}^p$ such that the following holds:*

1. *For $\bar{\mu}$-a.e. $\omega \in \Omega$, $g(\omega)$ is a limit point of $\{g_n(\omega)\}_{n=1}^\infty$.*
2. *For each $j \in J$, $\int_\Omega g d\mu_j \le a_j$.*
3. *For any $i \le p$ and $j \in J$ for which $\{g_n^i\}_{n=1}^\infty$ is uniformly μ_j-integrable, $\int_\Omega g^i d\mu_j = a_j^i$.*

Next we have a simple corollary of Case A of Theorem 2.4. It is a Fatou type lemma in terms of distributions that generalizes Proposition 3.12 in [**29**] from a scalar Loeb measure to the case of a vector Loeb measure.

For this and later results, we need to recall that a complete, separable metric space Z can be imbedded homeomorphically in a countable product of closed unit intervals; the closure of the image in that product is a compact metrizable space K, and Z (which we associate with its image) is a G_δ subset of K. (See Proposition 8.1.4 in [**7**].)

We shall refer to the topology of weak convergence of measures on Z. This is the topology on the Borel measures on Z generated by the bounded continuous real-valued functions, or equivalently (as shown in Theorem 6.1 of [**27**]) just the bounded uniformly continuous functions on Z. We shall implicitly refer to this topology when we say that a given sequence of measures converges weakly to a limit measure. Given an $\mathcal{A}$-measurable function f mapping Ω into Z and given $j \in J$, $\mu_j f^{-1}$ denotes the measure on Z taking the value $\mu_j(f^{-1}[B])$ at each Borel set $B \subseteq Z$.

COROLLARY 2.8. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of $\mathcal{A}$-measurable functions from Ω to a complete separable metric space Z with the property that for each $j \in J$, $\{\mu_j f_n^{-1}\}_{n=1}^\infty$ converges weakly to a Borel probability measure ν_j on Z. Then, there is an $\mathcal{A}$-measurable function f from Ω to Z such that $f(\omega)$ is a limit point of $\{f_n(\omega)\}_{n=1}^\infty$ for $\bar{\mu}$-a.e. $\omega \in \Omega$, and $\mu_j f^{-1} = \nu_j$.*

PROOF. Let K be the compact metric space described above in which Z is embedded as a dense G_δ-subspace. In applying Theorem 2.4, let Y be the space of continuous functions on K, and let X be the norm dual space of Y, which is the space of finite Borel measures on K. Any Borel measure on Z can also be viewed as a Borel measure on K by the trivial extension that gives measure zero to $K \setminus Z$.

The space K can be viewed as a topological subspace of X with respect to the weak* topology through the embedding $E(z) = \delta_z$ for each $z \in K$, where δ_z is the Dirac measure concentrated at the point z. Thus, $E(K)$ is a weak* compact set in X. Let $g_n(\omega) = \delta_{f_n(\omega)}$ for all $n \in \mathbb{N}$, $\omega \in \Omega$. Then, $\{g_n\}_{n=1}^\infty$ is a norm bounded, hence uniformly norm-integrable (with respect to each μ_j), sequence taking values in X. Also for each $n \in \mathbb{N}$, $j \in J$, the Gelfand integral $\int_\Omega g_n d\mu_j$ is simply $\mu_j f_n^{-1}$, and $\lim_{n\to\infty} \int_\Omega g_n d\mu_j = \nu_j$.

Now by Theorem 2.4, there is a function g from $(\Omega, \mathcal{A})$ to X such that $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^\infty$ for $\bar{\mu}$-a.e. $\omega \in \Omega$, and for each $j \in J$, g is Gelfand μ_j-integrable with $\int_\Omega g d\mu_j = \nu_j$. Since $\{g_n(\omega)\}_{n=1}^\infty$ is a sequence in the weak* compact set $E(K)$, its weak* limit point $g(\omega)$ must be in $E(K)$ as well. Hence, there is a measurable function f from $(\Omega, \mathcal{A})$ to K such that $f(\omega)$ is a limit point of $\{f_n(\omega)\}_{n=1}^\infty$ and $\mu_j f^{-1} = \nu_j$ for each $j \in J$.

As noted above, Z is a Borel, indeed a G_δ, set in K. For each $j \in J$, $\mu_j f^{-1}(Z) = \nu_j(Z) = 1$. Thus, $\bar{\mu} f^{-1}(K \setminus Z) = 0$. Fix $z_0 \in Z$.

By redefining the value of f on the $\bar{\mu}$-null set $f^{-1}(K \setminus Z)$ to be z_0, we obtain the desired result. □

EXAMPLE 2.9. In this example, we show that even for the case of a single measure, there may be no function g as described in Part B of Theorem 2.4 satisfying the first conclusion of that part (nor the last two) if the single measure is Lebesgue measure on the unit interval $[0,1]$. We will write $\int dt$ for the Lebesgue integral. For the space X, we take ℓ^p for $1 \leq p \leq \infty$. An example of Liapounoff (see [**8**], Page 262) constructs an $h : [0,1] \to \ell^p$ such that for no measurable subset $E \subset [0,1]$ is it true that for coordinate-wise integration, $\int_E h(t)\ dt = \frac{1}{2}\int_{[0,1]} h(t)\ dt$. For each $i \in \mathbb{N}$, let h^i be the i^{th} component of the function h. In the Liapounoff example, $h^1 \equiv 1$, and for $i > 1$, at any given $t \in [0,1]$, $h^i(t)$ is either 0 or $1/2^{i-1}$. For each $m \in \mathbb{N}$, let h_m be the function from $[0,1]$ into $\mathbb{R}^m$ with i^{th} component h^i for $1 \leq i \leq m$; let E_m be the measurable subset of $[0,1]$ given by the Theorem of Liapounoff such that $\int_{E_m} h_m(t)\ dt = \frac{1}{2}\int_{[0,1]} h_m(t)\ dt$, and let $F_m = [0,1] \setminus E_m$. Now for each $n \in \mathbb{N}$, let g_n be the function from $[0,1]$ into ℓ^p such that for each even $i \leq 2n$ the i^{th} component $g_n^i = h^{i/2}\mathbf{1}_{E_n}$, and for each odd $i \leq 2n-1$ the i^{th} component $g_n^i = h^{(i+1)/2}\mathbf{1}_{F_n}$; for each $i > 2n$, set $g_n^i = 0$. For example,

$$g_2 = \left\langle h^1\mathbf{1}_{F_2}, h^1\mathbf{1}_{E_2}, h^2\mathbf{1}_{F_2}, h^2\mathbf{1}_{E_2}, 0, 0, 0, \cdots \right\rangle,$$

$$\int_{[0,1]} g_2 = \left\langle \frac{1}{2}\int_{[0,1]} h^1, \frac{1}{2}\int_{[0,1]} h^1, \frac{1}{2}\int_{[0,1]} h^2, \frac{1}{2}\int_{[0,1]} h^2, 0, 0, 0, \cdots \right\rangle.$$

The sequence $\{g_n\}_{n=1}^{\infty}$ of uniformly norm bounded functions clearly satisfies the conditions of Theorem 2.4 with the obvious limit for the integrals. Suppose there is a function $g : [0,1] \to \ell^p$ with the properties of Part B of the theorem (we can assume that $g(t)$ is a weak* limit point of $\{g_n(t)\}_{n=1}^{\infty}$ for each $t \in [0,1]$). Fix an odd $i \in N$, say $i = 2m-1$; let s_i be the sequence that is 0 except at the i^{th} component where it is 1; fix a similar sequence s_{i+1} for $i+1 = 2m$. Testing with these two sequences in the predual of X, we see that for each $t \in [0,1]$, the i^{th} component $g^i(t) = 0$ or $g^i(t) = h^m(t)$; similarly $g^{i+1}(t) = 0$ or $g^{i+1}(t) = h^m(t)$. Moreover, $g^i(t) + g^{i+1}(t) = h^m(t)$. By the assumption with respect to g, we have

$$(1)\quad \int_{[0,1]} g^i(t)\ dt \leq \frac{1}{2}\int_{[0,1]} h^m(t)\ dt,\quad \int_{[0,1]} g^{i+1}(t)\ dt \leq \frac{1}{2}\int_{[0,1]} h^m(t)\ dt,$$

but of course

$$\int_{[0,1]} g^i(t)\ dt + \int_{[0,1]} g^{i+1}(t)\ dt = \int_{[0,1]} h^m(t)\ dt,$$

and so we have equality in both parts of Equation 1. Now for an element in the predual of X, we choose a sequence u for which each even component u^{2i} is $1/i^2$ and each odd component u^{2i-1} is 0. For each $n \in \mathbb{N}$ and $t \in [0,1]$, the value of $\langle g_n(t), u\rangle$ is either 0 or the sum $\sum_{i=1}^{n} h^i(t)/i^2$. It follows that for each $t \in [0,1]$, $\langle g(t), u\rangle$ is either 0 or $\sum_{i=1}^{\infty} h^i(t)/i^2$ (which is greater than 1). Now we have seen that for each $m \in \mathbb{N}$, either $g^{2m}(t) = h^m(t) \geq 0$, or $g^{2m}(t) = 0$. Therefore, if $\langle g(t), u\rangle = \sum_{i=1}^{\infty} h^i(t)/i^2$, we must have $g^{2m}(t) = h^m(t)$ for all $m \in \mathbb{N}$. Let E be the measurable subset of $[0,1]$ where $\langle g(t), u\rangle = \sum_{i=1}^{\infty} h^i(t)/i^2$. Using just the even components of $g(t)$, we have reached a contradiction, since they form the sequence $h\mathbf{1}_E$ with integral $\frac{1}{2}\int_{[0,1]} h(t)\ dt$.

Example 2.10. Let λ denote Lebesgue measure on $[0,1]$. We will modify here the previous example to show that the first two conclusions of Corollary 2.7 (and also the third) need not hold for multiples of λ. For that purpose, we use the sequence of measures $\mu_j = h^j\lambda$ (i.e., for each Borel set B in $[0,1]$, $\mu_j(B) = \int_B h^j d\lambda$), $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, μ_j is finite positive measure; let μ'_j be the probability measure normalized from μ_j. Our sequence of functions g_n will map $[0,1]$ into $\mathbb{R}^2$. Using the characteristic functions of the previous example, for each $n \in \mathbb{N}$, we set $g_n(t) = (\mathbf{1}_{F_n}(t), \mathbf{1}_{E_n}(t))$. For $n \geq j$ in $\mathbb{N}$ we have

$$\begin{aligned}
\int_{[0,1]} g_n(t)\, d\mu_j &= \left(\int_{F_n} h^j(t) d\lambda, \int_{E_n} h^j(t) d\lambda\right)\\
&= \left(\frac{1}{2}\int_{[0,1]} h^j(t) d\lambda, \frac{1}{2}\int_{[0,1]} h^j(t) d\lambda\right)\\
&= \left(\frac{1}{2}\mu_j([0,1]), \frac{1}{2}\mu_j([0,1])\right).
\end{aligned}$$

It follows that the sequence $\{g_n\}_{n=1}^{\infty}$ satisfies the conditions of Corollary 2.7 with the limit of the integrals for each probability measure μ'_j, $j \in \mathbb{N}$, equal to the point $\left(\frac{1}{2}\mu'_j([0,1]), \frac{1}{2}\mu'_j([0,1])\right) \in \mathbb{R}^2$. Now suppose there is a function $g : [0,1] \to \mathbb{R}^2$ as given by the corollary (we can assume that $g(t)$ is a weak* limit point of $\{g_n(t)\}_{n=1}^{\infty}$ for each $t \in [0,1]$). Given $n \in \mathbb{N}$, for each $t \in [0,1]$, $g_n(t) = (1,0)$ or $(0,1)$ in $\mathbb{R}^2$, so the same is true for $g(t)$. That is, there are disjoint measurable sets E and

F with union $[0,1]$ such that $g = (\mathbf{1}_F, \mathbf{1}_E)$. Given $j \in \mathbb{N}$,

$$\int_{[0,1]} g(t) d\mu'_j = (\mu'_j(F), \mu'_j(E)).$$

By the assumption on g,

$$\mu'_j(F) \leq \frac{1}{2}\mu'_j([0,1]),\ \mu'_j(E) \leq \frac{1}{2}\mu'_j([0,1]). \tag{2}$$

Since $\mathbf{1}_F + \mathbf{1}_E \equiv 1$ on $[0,1]$, the two inequalities in Equation 2 are actually equalities, which implies that for each $j \in \mathbb{N}$, $\int_E h^j(t) d\lambda = \frac{1}{2}\int_{[0,1]} h^j(t) d\lambda$. We have reached a contradiction, since by the Liapounoff example, no such a measurable set E can exist. Note that by the classical Fatou Lemma, one cannot get a similar contradiction for all of Corollary 2.7 with just a sequence of mappings into $\mathbb{R}$; the equality assertion still fails, however, for the sequence $\{\mathbf{1}_{F_n}\}_{n=1}^{\infty}$.

Example 2.11. Let $(\Omega, \mathcal{A}, P)$ be any atomless probability space, not just a Loeb probability space. For each $i \in \mathbb{N}$, let s_i be the sequence that is 1 at the i^{th}-coordinate and 0 at all the others. We form a sequence of functions $g_n : \Omega \to \ell^p$ for $2 \leq p \leq \infty$ as follows. For each $n \in \mathbb{N}$, form a measurable partition of Ω into n disjoint subsets each of measure $1/n$. Order the subsets, and for all ω in the j^{th} subset, $1 \leq j \leq n$, let $g_n(\omega) = \sqrt{n} s_j$. At each $\omega \in \Omega$, it is easy to see that $\|g_n(\omega)\|_p = \sqrt{n}$. So the sequence $\{g_n\}_{n=1}^{\infty}$ is not weak* tight. However, for $p = 2$,

$$\left\| \int g_n(\omega) dP \right\|_2 = \sqrt{\sum_{j=1}^{n} \left(\frac{\sqrt{n}}{1} \cdot \frac{1}{n} \right)^2} = 1,$$

and for $p > 2$, each integral has an even smaller norm. Thus, the sequence of integrals has a weak*-convergent subsequence.

3. Distribution of Correspondences on Vector Loeb Spaces

In this section we present some properties of the distribution of correspondences induced by vector Loeb measures. We recall some basic notions first.

Let Λ and W be nonempty sets, and $\mathcal{P}(W)$ the power set of W. A mapping from Λ to $\mathcal{P}(W) \setminus \{\emptyset\}$ is called a correspondence from Λ to W.

We will work with the same Loeb measurable space $(\Omega, \mathcal{A})$ supplied with the Loeb probability measures $\overline{\mu}$ and μ_j, $j \in J$ as in Section 2. Let F be a correspondence from $(\Omega, \mathcal{A})$ to a complete separable metric space X with its Borel σ-algebra $\mathcal{B}(X)$. The correspondence

F is said to be measurable if for each close subset C of X, the set $\{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\}$ is measurable in $\mathcal{A}$. The correspondence F is said to be closed (compact) valued if $F(\omega)$ is a closed (compact) subset of X for each $\omega \in \Omega$. A measurable function f from $(\Omega, \mathcal{A})$ to X is said to be a measurable selection of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

Let $\mathcal{M}(X)$ be the space of Borel probability measures on X endowed with the topology of weak convergence of measures, $(\mathcal{M}(X))^J$ the product space of $|J|$ copies of $\mathcal{M}(X)$ with the usual product topology. Recall that for a measurable mapping φ from a probability space $(\Omega, \mathcal{A}, \nu)$ to X, we use $\nu\varphi^{-1}$ to denote the Borel probability measure on X induced by φ; this measure is often called the distribution of φ. We also use $\mu\varphi^{-1}$ to denote $\{\mu_j\varphi^{-1}\}_{j\in J}$, which belongs to $(\mathcal{M}(X))^J$.

For a correspondence F, define

$$\mathcal{D}_F = \{\mu\varphi^{-1} : \varphi(\cdot) \text{ is a measurable selection from } F\}.$$

As noted in the second paragraph of Section 2, we can assume that $\mathcal{A}$ is complete with respect to $\bar{\mu}$; thus the classical Kuratowski-Ryll-Nardzewski Theorem [**1**, p.505] implies that in this case, a measurable correspondence F has a measurable selection (i.e., $\mathcal{D}_F \neq \emptyset$).

The following results state some fundamental properties about the distribution of correspondences induced by vector Loeb measures, which generalize Theorems 1, 2, 4 and 5 in [**29**] from a scalar Loeb measure to the case of vector Loeb measures. In [**10**], the same type of results are considered for the simple case that X is a finite set while the relevant vector measures are not necessarily vector Loeb measures.

PROPOSITION 3.1. *(1) If F is closed valued, then $\mathcal{D}_F$ is closed.*

(2) If F is compact valued, then $\mathcal{D}_F$ is compact.

(3) If μ_j is an atomless Loeb probability space for each $j \in J$, then $\mathcal{D}_F$ is convex.

PROOF. (1) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable selections of F such that for each $j \in J$, $\mu_j f_n^{-1}$ converges to a Borel probability measure ν_j in topology of weak convergence of measures on X. By Corollary 2.8, there is a measurable function f from $(\Omega, \mathcal{A})$ to X such that $f(\omega)$ is a limit point of $\{f_n(\omega)\}_{n=1}^{\infty}$, $\bar{\mu}$-a.e. $\omega \in \Omega$, and $\mu_j f^{-1} = \nu_j$ for any $j \in J$. After redefining the value of f on a $\bar{\mu}$-null set, we see that f is a measurable selection of F, and $\nu = \mu f^{-1} \in \mathcal{D}_F$, whence $\mathcal{D}_F$ is closed.

(2) Fix any $j \in J$. It follows from Theorem 4 in [**29**] that the set $\mathcal{D}_F^j = \{\mu_j\varphi^{-1} : \varphi(\cdot)$ is a measurable selection from $F\}$ is compact. Hence, $\prod_{j\in J} \mathcal{D}_F^j$ is compact in $(\mathcal{M}(X))^J$. Since $\mathcal{D}_F$ is closed by (1),

the compactness of $\mathcal{D}_F$ follows from that fact that $\mathcal{D}_F$ is a closed subset of $\prod_{j\in J}\mathcal{D}_F^j$.

(3) let $c\in[0,1]$ and φ and $\tilde{\varphi}$ be two measurable selections from F. Let $\tau = c\,\mu\varphi^{-1} + (1-c)\,\mu\tilde{\varphi}^{-1}$. Define a mapping $f:\Omega\to\mathcal{M}(X)$ by letting $f(\omega) = c\delta_{\varphi(\omega)} + (1-c)\delta_{\tilde{\varphi}(\omega)}$, where δ_x is the Dirac measure at x for $x\in X$. Then $\tau(\{B\}) = \int_\Omega f(\omega)(B)d\mu(\omega)$ for any Borel set B in X. It follows from the proof of Corollary 2.6 in [**22**] that there exists a measurable function $\psi:\Omega\to X$ such that $\tau = \mu\psi^{-1}$ and $\psi(\omega)\in\{\varphi(\omega),\tilde{\varphi}(\omega)\}$ for all $\omega\in\Omega$. Since $\{\varphi(\omega),\tilde{\varphi}(\omega)\}\subseteq F(\omega)$, this implies that ψ is a measurable selection of F. Hence $\tau\in\mathcal{D}_F$, and $\mathcal{D}_F$ is convex. $\square$

Next, let G be a correspondence from a topological space Y to another topological space Z. Let y_0 be a point in Y. Then G is said to be upper semicontinuous at y_0 if for any open set U that contains $G(y_0)$, there exists a neighborhood V of y_0 such that for each $y\in V$, $G(y)\subseteq U$. G is said to be upper semicontinuous on Y if it is upper semicontinuous at every point $y\in Y$.

PROPOSITION 3.2. *Let X be a complete separable metric space, Y a metric space, and F a closed valued correspondence from $\Omega\times Y$ to X. For each fixed $y\in Y$, let F_y denote the correspondence $F(\cdot,y)$ from Ω to X, which is assumed to be measurable. Let G be a compact-valued correspondence from Ω to X such that $F(\omega,y)\subseteq G(\omega)$ for each $\omega\in\Omega$ and $y\in Y$. If the correspondence $F(\omega,\cdot)$ is upper semicontinuous on Y for each fixed $\omega\in\Omega$, then $\mathcal{D}_{F_y}$ is upper semicontinuous in $y\in Y$.*

PROOF. Since the correspondence G is compact valued, $\mathcal{D}_G$ is compact by Proposition 3.1. By assumption, G dominates F_y for all $y\in Y$; therefore, $\mathcal{D}_{F_y}$ induces a closed valued correspondence from Y to the compact space $\mathcal{D}_G$. Note that a closed valued correspondence from a metrizable space to a compact metrizable space is upper semicontinuous if and only if it has a closed graph (see Proposition 1.4.8 on p.42 of [**2**]). Therefore the result on upper semicontinuity follows from Corollary 2.8. $\square$

REMARK 3.3. Examples 1 and 3 in [**29**] show that the results in Propositions 3.1 and 3.2 can fail to hold if the vector Loeb space is replaced by the Lebesgue unit interval.

4. Finite Games with Private and Public Information

As an application of the results in Section 3, we establish in this section the existence of a pure strategy equilibrium for finite games with both private and public information and with compact metric action spaces.

A *game with private and public information* Γ consists of a finite set of ℓ players and the following associated spaces and functions. Each player i chooses actions from a compact metric space A_i; the product $\Pi_{j=1}^{\ell} A_j$ is denoted by A. For each player i, a measurable space $(T_i, \mathcal{T}_i)$ represents the personal information and events known to the player but not necessarily to the other players, such as evolving, privately obtained information about the management of various companies. A finite or countably infinite set $T_0 = \{t_{0k} : k \in K\}$ represents those states that are to be publicly announced to all the players; let $\mathcal{T}_0$ be the power set of T_0. Another finite or countably infinite set $S_0 = \{s_{0q} : q \in Q\}$ represents the payoff-relevant common states that affect the payoffs of all the players with $\mathcal{S}_0$ the power set of S_0. The payoff for the i-th player depends on the actions chosen by all the players, and player i's private information $t_i \in T_i$, together with a payoff relevant common state $s_0 \in S_0$. That is, the i-th player's payoff is given by a function $u_i : A \times S_0 \times T_i \longrightarrow \mathbb{R}$; we assume that for each $a \in A$ and $s_0 \in S_0$, $u_i(a, s_0, \cdot)$ is $\mathcal{T}_i$-measurable on T_i. The product measurable space $(\Omega, \mathcal{F}) := (S_0 \times \Pi_{j=0}^{\ell} T_j, \mathcal{S}_0 \times \Pi_{j=0}^{\ell} \mathcal{T}_j)$ equipped with a probability measure η constitutes the information space of the game Γ. We assume that there is an integrable function ψ on $(\Omega, \mathcal{F}, \eta)$ such that for each payoff function u_i, $u_i(a, s_0, t_i) \leq \psi(s_0, t_0, t_1, \cdots, t_l)$ holds for each $a \in A$, and each $(s_0, t_0, t_1, \cdots, t_l) \in \Omega$. We also assume that each payoff $u_i(\cdot, s_0, t_i)$ is a continuous function on A when s_0 and t_i are fixed.

The players can use their private information as well as the publicly announced information. Thus, a *pure strategy* for player i is a measurable mapping from $T_0 \times T_i$ to A_i. A *pure strategy profile* is a collection $g = (g_1, \ldots, g_l)$ of pure strategies that specify a pure strategy for each player. In what follows, when i is given, we shall abbreviate a product over all indices $1 \leq j \leq \ell$ except for $j = i$ by $\Pi_{j \neq i}$; i.e., $\Pi_{j \neq i}$ means $\Pi_{1 \leq j \leq \ell, j \neq i}$. For each player $i = 1, \ldots, l$, we shall use the following (conventional) notation: $A_{-i} = \Pi_{j \neq i} A_j$. For a in the product $A = \Pi_{j=1}^{\ell} A_j$, we write a_{-i} for the projection of a into A_{-i}; we also denote a with the pair (a_i, a_{-i}). Similarly, for a pure strategy profile $g = (g_1, \ldots, g_l)$, we write $g = (g_i, g_{-i})$.

If the players play a pure strategy profile $g = (g_1, \ldots, g_l)$, the resulting *expected payoff* for player i can be written as

$$U_i(g) = U_i(g_1, \ldots, g_l) = \int_{\Omega} u_i(g_1(t_0, t_1), \ldots, g_l(t_0, t_l), s_0, t_i) d\eta. \tag{3}$$

A *pure strategy equilibrium* for Γ is a pure strategy profile $g^* = (g_1^*, \ldots, g_l^*)$, such that for each $i = 1, \ldots, l$, $U_i(g_i^*, g_{-i}^*) \geq U_i(g_i, g_{-i}^*)$ for any other pure strategy g_i player i can choose.

Let η_0 be the marginal probability measure on the countable set $S_0 \times T_0$. For simplicity, we denote $\eta_0(\{(t_{0k}, s_{0q})\})$ by α_{kq}. For the principal result of this section, we will need a condition on the probability measure η. For each given $t_{0k} \in T_0$ and $s_{0q} \in S_0$, let η^{kq} denote the conditional probability measure of η on the space $(\Pi_{j=1}^l T_j, \Pi_{j=1}^l \mathcal{T}_j)$; such a conditional probability measure always exists since both S_0 and T_0 are countable. For each player $i = 1, \ldots, l$, let τ_i and μ_i^{kq} be the respective marginal measures of η and η^{kq} on the space $(T_i, \mathcal{T}_i)$. The players' private information is said to be conditionally independent, given the public and payoff-relevant common information if for each $k \in K$, $q \in Q$,

$$\eta^{kq} = \Pi_{i=1}^{\ell} \mu_i^{kq}. \tag{4}$$

Now we can apply the results of Section 3 to obtain the existence of a pure strategy equilibrium for the game Γ.

THEOREM 4.1. *Assume that (1) the players' private information is conditionally independent, given the public and payoff-relevant common information; (2) the marginal probability measure τ_i of η on $(T_i, \mathcal{T}_i)$ is an atomless Loeb measure. Then there exists a pure strategy equilibrium for the game Γ.*

PROOF. Fix player i. Recall that the conditional probability measure μ_i^{kq} is uniquely defined only when $\alpha_{kq} > 0$. When $\alpha_{kq} = 0$, we can redefine μ_i^{kq} to be τ_i without loss of generality. It is clear that for each $C_i \in \mathcal{T}_i$, $\tau_i(C_i) = \sum_{k \in K,\, q \in Q} \alpha_{kq} \mu_i^{kq}(C_i)$. Thus, for the case that $\alpha_{kq} > 0$, μ_i^{kq} is absolutely continuous with respect to the atomless Loeb measure τ_i.

Hence, we can assume that for each $k \in K$, $q \in Q$, μ_i^{kq} is atomless and absolutely continuous with respect to the Loeb measure τ_i. Let β_i^{kq} be the Radon-Nikodym derivative of μ_i^{kq} with respect to τ_i, which is Loeb integrable on the Loeb measure space $(T_i, \mathcal{T}_i, \tau_i)$. By taking an internal S-integrable lifting of β_i^{kq} and using it as an internal density, we know that μ_i^{kq} is a Loeb measure on $(T_i, \mathcal{T}_i)$ as well.

As in Section 3, the space $\mathcal{M}(A_i)$ of Borel probability measures is endowed with the topology of weak convergence of measures. Let $[\mathcal{M}(A_i)]^Q$ be the product space of $|Q|$ copies of $\mathcal{M}(A_i)$ with the product topology, which as a countable product of compact metrizable spaces, can be given a compatible metric to make it a compact metric space.

Let $g = (g_1, \ldots, g_l)$ be a pure strategy profile. Denote $g_i(t_{0k}, t_i)$ by $g_i^k(t_i)$ for $k \in K$. Thus, for each $k \in K$, g_i^k is a measurable mapping from T_i to A_i. With the assumption of conditional independence in part (1) of Theorem 4.1, we can rewrite player i's payoff in Equation (3) as

$$\begin{aligned}
U_i(g) &= \sum_{k \in K} \sum_{q \in Q} \alpha_{kq} \int_{\Pi_{j=1}^l T_j} u_i(g_1^k(t_1), \ldots, g_l^k(t_l), s_{0q}, t_i) d\eta^{kq} \\
&= \sum_{k \in K} \sum_{q \in Q} \alpha_{kq} \int_{T_i \times \Pi_{j \neq i} T_j} u_i(g_1^k(t_1), \ldots, g_l^k(t_l), s_{0q}, t_i) d\left(\mu_i^{kq} \times \Pi_{j \neq i} \mu_j^{kq}\right) \\
&= \sum_{k \in K} \sum_{q \in Q} \alpha_{kq} \int_{T_i \times A_{-i}} u_i(g_i^k(t_i), a_{-i}, s_{0q}, t_i) d\left(\mu_i^{kq} \times \Pi_{j \neq i} \mu_j^{kq} (g_j^k)^{-1}\right) \\
&= \sum_{k \in K} \int_{T_i} \sum_{q \in Q} \alpha_{kq} \beta_i^{kq}(t_i) \int_{A_{-i}} u_i(g_i^k(t_i), a_{-i}, s_{0q}, t_i) d\left(\Pi_{j \neq i} \mu_j^{kq} (g_j^k)^{-1}\right) d\tau_i.
\end{aligned}$$

The above equation says that player i's payoff only depends on the conditional distributions of the other players' strategies on their action spaces, given the public information t_{0k}, $k \in K$, and payoff-relevant common information s_{0q}, $q \in Q$.

For convenience, we introduce the following notation. Let $\gamma = (\gamma_1, \ldots, \gamma_l) \in \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$. Then, for any player j, $\gamma_j = (\gamma_j^q)_{q \in Q} \in (\mathcal{M}(A_j))^Q$ can be interpreted as a conditional distribution for player j's strategy given the public information t_{0k} and payoff-relevant common information s_{0q}. In addition, γ_{-i} specifies the conditional distributions for all the players except for player i.

This suggests that we consider the following function

$$\begin{aligned}
V_i^{k\gamma}(g_i^k) &= \int_{T_i} \sum_{q \in Q} \alpha_{kq} \beta_i^{kq}(t_i) \int_{A_{-i}} u_i(g_i^k(t_i), a_{-i}, s_{0q}, t_i) d\Pi_{j \neq i} \gamma_j^q(a_{-i}) d\tau_i \\
&= \int_{T_i} w_i^k(g_i^k(t_i), t_i, \gamma) d\tau_i(t_i),
\end{aligned}$$

where

$$w_i^k(a_i, t_i, \gamma) = \sum_{q \in Q} \alpha_{kq} \beta_i^{kq}(t_i) \int_{A_{-i}} u_i(a_i, a_{-i}, s_{0q}, t_i) d\Pi_{j \neq i} \gamma_j^q(a_{-i}).$$

Note that the function $V_i^{k\gamma}(g_i^k)$ is actually independent of γ_i. However, it is more convenient to take the whole γ as a parameter, and we do so. It is clear that given γ_{-i} as the conditional distributions for all the players except for player i, player i should choose a measurable response function $g_i^k : T_i \to A_i$ to maximize $V_i^{k\gamma}(g_i^k)$ for each state of public information $t_0 = t_{0k}$, $k \in K$.

It is obvious that for each fixed $a_i \in A_i$, $\gamma \in \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$, $w_i^k(a_i, t_i, \gamma)$ is $\mathcal{T}_i$-measurable with respect to $t_i \in T_i$. For any $t_i \in T_i$, the fact that $w_i^k(a_i, t_i, \gamma)$ is continuous on $A_i \times \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$ follows from the Stone-Weierstrass theorem, as pointed out in Section 4.3 of [**19**].

For any $t_i \in T_i$, $\gamma \in \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$, let

$$\Phi_i^k(t_i, \gamma) = \left\{a_i \in A_i : w_i^k(a_i, t_i, \gamma) \geq w_i^k(x, t_i, \gamma) \ \ \forall x \in A_i\right\}.$$

For each fixed $t_i \in T_i$, the correspondence $\Phi_i^k(t_i, \cdot)$ is upper semicontinuous on $\Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$ by Berge's maximum theorem (see, e.g., [**1**, p.473]). For any fixed $\gamma \in \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$, the correspondence $\Phi_i^k(\cdot, \gamma)$ is measurable by Theorem 14.91 in [**1**, p.508].

We shall now show the existence of a pure strategy equilibrium for the game Γ. Denote the vector measure $(\mu_i^{kq})_{q \in Q}$ by μ_i^k. For any $k \in K$ and $\gamma \in \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$, let

$$G_i^k(\gamma) = \{\mu_i^k \varphi^{-1} : \varphi(\cdot) \text{ is a measurable selection of } \Phi_i^k(\cdot, \gamma)\},$$

and $G^k(\gamma) = \Pi_{i=1}^l G_i^k(\gamma)$. It follows from Propositions 3.1 and 3.2 that for each i, $G_i^k(\cdot)$ is convex and compact valued, and upper-semicontinuous on $\Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$, so is the product $G^k(\gamma)$. We can apply the Fixed Point Theorem of Fan [**9**] and Glicksberg [**11**] to assert the existence of a $\gamma^{*k} = (\gamma_1^{*k}, \ldots, \gamma_l^{*k}) \in \Pi_{j=1}^l \left[(\mathcal{M}(A_j))^Q\right]$ such that $\gamma^{*k} \in G^k(\gamma^{*k})$. This means that for each player i and each k, $\gamma_i^{*k} \in G_i^k(\gamma^{*k})$, i.e., there exists a measurable selection g_i^{*k} of the correspondence $\Phi_i^k(\cdot, \gamma^{*k})$ such that $\mu_i^k \left(g_i^{*k}\right)^{-1} = \gamma_i^{*k}$. This means that for any player i and $k \in K$, g_i^{*k} maximizes $V_i^{k\gamma^{*k}}(\cdot)$ on the space of measurable mappings from T_i to A_i. Therefore, the pure strategy profile $g^* = (g_1^*, \ldots, g_l^*)$ is a pure strategy equilibrium for the game Γ. □

Remark 4.2. (1) Finite games with both private and public information are introduced in [**10**] for the case of finite actions. The existence of a pure strategy equilibrium is shown there for games with finite actions, where the relevant probability measures are not necessarily Loeb measures. The assumption of finite actions in games is a serious restriction since many games require infinite action spaces. This paper is the first one to study games with general compact action spaces and both private and public information.

(2) As shown in [**17**], the existence result of a pure strategy equilibrium can fail to hold for games with general compact metric action

spaces and with a Lebesgue type information space. Thus, Theorem 4.1 may fail to hold if the private information spaces $(T_i, \mathcal{T}_i, \tau_i)$ are not assumed to be Loeb spaces.

(3) In [**10**], each player's private information is also divided into two parts, one for payoff-relevant private information and another for strategy-relevant private information. Using the techniques in [**10**] and [**18**], we can generalize the result in Theorem 4.1 to this more general setting.

5. Proof of Theorem 2.4

The following lemma shows that if the internal probability measure induced by an internal mapping h is in the weak monad of a standard tight measure γ on a separable metric space, then h is near standard and its standard part $^\circ h$ on the corresponding Loeb space induces γ.

LEMMA 5.1. *Let X be a standard, separable metric space with metric ρ and the Borel σ-algebra $\mathcal{B}$, and let γ be a standard tight probability measure on $(X, \mathcal{B})$. Let P_0 be an internal probability measure on $(\Omega, \mathcal{A}_0)$, and $(\Omega, \mathcal{A}, P)$ the corresponding Loeb space. Let h be an internal, measurable map from $(\Omega, \mathcal{A}_0)$ to $(^*X, ^*\mathcal{B})$, and ν an internal probability measure on $(^*X, ^*\mathcal{B})$ such that $\nu = P_0 h^{-1}$, and $\nu \simeq {}^*\gamma$ in the nonstandard extension of the topology of weak convergence of Borel measures on X. Fix $x_0 \in X$. The standard part $^\circ h(\omega)$ exists for P-almost all $\omega \in \Omega$; where $h(\omega)$ is not near-standard, set $^\circ h(\omega) = x_0$. Now for this function $^\circ h$, $\gamma = P(^\circ h)^{-1}$.*

PROOF. By definition, for every standard, bounded, continuous real-valued function f on X,

$$\int_{^*X} {}^*f \, d\nu \simeq \int_{^*X} {}^*f \, d^*\gamma = \int_X f \, d\gamma.$$

Let K_0 be a compact subset of X; it follows from the tightness of γ that for each $n \in \mathbb{N}$, one can let K_n be a compact subset of X such that $\gamma(K_n) > 1 - \frac{1}{2n}$, and $K_n \supseteq K_{n-1}$. Given K_n, and $j \in \mathbb{N}$, the set

$$V_n^j := \{x \in X : \rho(x, K_n) < \frac{1}{j}\}$$

has the property that $\nu(^*V_n^j) > 1 - \frac{1}{n}$ since there is a continuous function $0 \le f \le 1$ on X that is 1 on K_n and 0 off of V_n^j, whence

$$\nu(^*V_n^j) = \int_{^*V_n^j} 1 \, d\nu \ge \int_{^*V_n^j} {}^*f \, d\nu \simeq \int_{V_n^j} f \, d\gamma \ge \gamma(K_n) > 1 - \frac{1}{2n}.$$

Now for each $n \in \mathbb{N}$, the monad $m(K_n) = \cap_{j \in \mathbb{N}} {}^*V_n^j$, and

$$h^{-1}[m(K_n)] = h^{-1}\left[\cap_{j \in \mathbb{N}} {}^*V_n^j\right] = \cap_{j \in \mathbb{N}} h^{-1}\left[{}^*V_n^j\right]$$

is externally measurable in Ω. Moreover, $P\left(h^{-1}[m(K_n)]\right) \geq 1 - \frac{1}{n}$.

Given $n \in \mathbb{N}$, ${}^\circ h$ is defined on $h^{-1}[m(K_n)]$; it takes its values in K_n and is measurable: A modification of the well-known proof of measurability starts with an open set $O \subseteq X$, with $O \cap K_n \neq \varnothing$, and a finite or countably infinite dense subset $\{x_i\}$ of $O \cap K_n$. For each x_i, we choose a radius r_i so that the open ball $B(x_i, r_i) \subset O$. The set

$$E := \cup_i h^{-1}\left[{}^*B(x_i, r_i/2) \cap m(K_n)\right]$$

is externally measurable in Ω. Moreover, if $\omega \in E$, then for some $i \in \mathbb{N}$,

$$h(\omega) \in {}^*B(x_i, r_i/2) \cap m(K_n) \Rightarrow {}^\circ h(\omega) \in B(x_i, r_i) \cap K_n \Rightarrow {}^\circ h(\omega) \in O \cap K_n.$$

On the other hand, if $\omega \in \Omega$ and ${}^\circ h(\omega) \in O \cap K_n$, then for some $i \in \mathbb{N}$,

$${}^\circ h(\omega) \in B(x_i, r_i/4) \cap K_n \Rightarrow h(\omega) \in {}^*B(x_i, r_i/2) \cap m(K_n) \Rightarrow \omega \in E.$$

It follows that $E = {}^\circ h^{-1}[O \cap K_n]$.

Now, ${}^\circ h$ defines a measurable mapping from the externally measurable set $\cup_n h^{-1}[m(K_n)]$ to $\cup_n K_n$. Since $P\left(\cup_n h^{-1}[m(K_n)]\right) = 1$, one can define ${}^\circ h$ on the complement of $\cup_n h^{-1}[m(K_n)]$ in Ω to be a constant $x_0 \in X$. With this extension, ${}^\circ h$ is a measurable mapping defined on $(\Omega, \mathcal{A}, P)$.

Finally, given a bounded, continuous, real-valued function f on X,

$$\begin{aligned}\int_X f \, dP({}^\circ h)^{-1} &= \int_\Omega f \circ {}^\circ h \, dP = \int_\Omega \mathrm{st}\left({}^*f \circ h\right) \, dP \\ &\simeq \int_\Omega {}^*f \circ h \, dP_0 = \int_{{}^*X} {}^*f \, d\nu \simeq \int_{{}^*X} {}^*f \, d^*\gamma = \int_X f \, d\gamma.\end{aligned}$$

It follows that $\gamma = P\left({}^\circ h\right)^{-1}$ on X. □

The next lemma shows a way of identifying the limit points for a sequence of functions.

Lemma 5.2. *Let X be a standard, separable metric space with metric ρ and the Borel σ-algebra $\mathcal{B}$. Let P_0 be an internal probability measure on $(\Omega, \mathcal{A}_0)$, and $(\Omega, \mathcal{A}, P)$ the corresponding Loeb space. Fix an internal sequence $\{h_n : n \in {}^*\mathbb{N}\}$ of measurable maps from $(\Omega, \mathcal{A}_0)$ to $({}^*X, {}^*\mathcal{B})$. Given a nonempty compact subset K of X, there is an unlimited $H_K \in {}^*\mathbb{N}$ and a P-null set $S_K \subset \Omega$ such that for each unlimited $n \in {}^*\mathbb{N}$ with $n \leq H_K$, h_n has the property that if $h_n(\omega)$ has standard part in K and $\omega \notin S_K$, then for any standard $\varepsilon > 0$, there are infinitely many limited $k \in \mathbb{N}$ for which ${}^*\rho(h_k(\omega), h_n(\omega)) < \varepsilon$.*

PROOF. Given $l \in \mathbb{N}$, cover K with a finite number, n_l, of open balls of radius $1/l$. Let $B(l,j)$, $1 \le j \le n_l$ denote the nonstandard extension of the j^{th} ball. For each $i \in {}^*\mathbb{N}$, set $A_i(l,j) := \{\omega \in \Omega : h_i(\omega) \notin B(l,j)\}$. For each $k \in \mathbb{N}$, choose an unlimited $m_k(l,j) \in {}^*\mathbb{N}$ so that

$$P\left(\cap_{i=k}^{m_k(l,j)} A_i(l,j)\right) = P\left(\cap_{i=k,i\in\mathbb{N}}^{\infty} A_i(l,j)\right).$$

Set

$$S_k(l,j) := \left(\cap_{i=k,i\in\mathbb{N}}^{\infty} A_i(l,j)\right) \setminus \cap_{i=k}^{m_k(l,j)} A_i(l,j).$$

Let H_K be an unlimited element of ${}^*\mathbb{N}$ such that H_K is smaller than $m_k(l,j)$ for every $l \in \mathbb{N}$, every $j \le n_l$, and every $k \in \mathbb{N}$. Let S_K be the P-null set formed by the union of the sets $S_k(l,j)$ for every $l \in \mathbb{N}$, every $j \le n_l$, and every $k \in \mathbb{N}$. To see that H_K works, fix an unlimited $n \in {}^*\mathbb{N}$ with $n \le H_K$. Suppose that $h_n(\omega)$ has a standard part in K but for some $l \in \mathbb{N}$, there are at most finitely many limited $k \in \mathbb{N}$ for which ${}^*\rho(h_k(\omega), h_n(\omega)) < 2/l$. Then for some $j \le n_l$, $h_n(\omega) \in B(l,j)$, and by assumption there is a limited $k \in \mathbb{N}$ such that for all limited $i \ge k$, $h_i(\omega) \notin B(l,j)$. It follows that $\omega \in S_k(l,j) \subseteq S_K$. $\square$

In stating Case B of Theorem 2.4 we assume either that X has the norm approximation property or, if that assumption does not hold, that the sequence $\{g_n\}_{n=1}^{\infty}$ is weak* μ_j-tight. We now show that the validity of the first assumption implies that of the second.

PROPOSITION 5.3. *Suppose that in Case B of Theorem 2.4, X has the norm approximation property. It then follows that for each $j \in J$, the sequence $\{g_n\}_{n=1}^{\infty}$ is weak* μ_j-tight.*

PROOF. Assume there is an increasing sequence y_m of nonnegative elements in Y with $\lim_{m\to\infty} \langle x, y_m\rangle = \|x\|$. Fix $j \in J$. We claim that for each $n \in \mathbb{N}$, $\|g_n\|$ is μ_j-integrable and

$$\int_\Omega \|g_n\|\, d\mu_j = \left\|\int_\Omega g_n\ d\mu_j\right\|.$$

To show this we use an argument that will appear in [**24**] by Heinrich Lotz. Given the sequence y_m, by the Monotone Convergence Theorem,

$$\left\|\int_\Omega g_n d\mu_j\right\| = \lim_{m\to\infty} \left\langle \int_\Omega g_n d\mu_j, y_m\right\rangle = \lim_{m\to\infty} \int_\Omega \langle g_n, y_m\rangle\, d\mu_j$$
$$= \int_\Omega \lim_{m\to\infty} \langle g_n, y_m\rangle\, d\mu_j = \int_\Omega \|g_n\|\, d\mu_j.$$

Now, since the Gelfand integrals $\int_\Omega g_n d\mu_j$ of the functions g_n converge in the weak*-topology, they are uniformly norm-bounded by the Uniform Boundedness Principle. Therefore, we may assume that for some

$M_j > 0$ and all $n \in \mathbb{N}$, $\left\| \int_\Omega g_n \, d\mu_j \right\| \le M_j$. Therefore, for each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$,

$$\int_{\{\|g_n(\omega)\| \ge k\}} \|g_n\| \, d\mu_j \le M_j,$$

whence

$$\mu_j \left(\{\omega \in \Omega : \|g_n(\omega)\| \ge k\}\right) \le M_j/k,$$

and so the sequence $\{g_n\}_{n=1}^\infty$ is weak* μ_j-tight. □

Proof of Theorem 2.4. We begin with facts that are valid for either Case A or B. We introduce a special symbol $\bar{j} \notin J$ and denote $\bar{\mu}$ by $\mu_{\bar{j}}$; let $\bar{J} = J \cup \{\bar{j}\}$.

Given Proposition 5.3, it follows from our assumptions that the sequence $\{g_n\}_{n=1}^\infty$ is weak* μ_j-tight for each $j \in J$, and it is easy to see that it is also weak* $\bar{\mu}$-tight. By the Uniform Boundedness Theorem, any weak* compact set is norm-bounded. Hence, for each $j \in \bar{J}$ it follows from the weak* μ_j-tightness condition on $\{g_n\}_{n=1}^\infty$ that

$$\varepsilon_{kj} := \sup_{n \ge 1} \mu_j(\{\omega \in \Omega : \|g_n(\omega)\| > k\}) \tag{5}$$

goes to zero as k goes to infinity.

Let $\{y_m\}_{m=1}^\infty$ be a countable dense set in the closed unit ball of Y. We use the sequence $\{y_m\}_{m=1}^\infty$ to define a metric ρ on X as follows: For $x_1, x_2 \in X$, $\rho(x_1, x_2) = \sum_{m=1}^\infty \frac{1}{2^m} |\langle x_1 - x_2, y_m \rangle|$. The topology induced by ρ agrees with the weak* topology on all the weak* compact sets of X; the two topologies also generate the same Borel σ-algebra on X.

By the classical Prohorov Theorem, the tight sequence of measures $\{\mu_j {g_n}^{-1}\}_{n=1}^\infty$ has a convergent subsequence in the topology of weak convergence of Borel measures on (X, ρ). Since $\bar{J}$ is finite or countably infinite, a standard diagonalization argument shows that one can choose a subsequence $\{g_{k_q}\}_{q=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ such that for all $j \in \bar{J}$, $\mu_j (g_{k_q})^{-1}$ converges weakly to some (tight) Borel probability measure γ_j on (X, ρ). The conclusions of Theorem 2.4 still holds even if one works with just a subsequence of the relevant sequence of functions. Without loss of generality, therefore, we assume that the subsequence $\{g_{k_q}\}_{q=1}^\infty$ is the whole sequence $\{g_n\}_{n=1}^\infty$. The measure $\mu_j g_n^{-1}$ now converge weakly to γ_j on (X, ρ).

Next, for each $n \in \mathbb{N}$ let $A_1^n = \{\omega \in \Omega : \|g_n(\omega)\| \le 1\}$, and let $A_k^n = \{\omega \in \Omega : k - 1 < \|g_n(\omega)\| \le k\}$ for $k \ge 2$. The sequence $\{A_k^n, k \in \mathbb{N}\}$ forms an $\mathcal{A}$-measurable partition of Ω. We can find a sequence of internal sets Ω_k^n, $k \in \mathbb{N}$, such that $\bar{\mu}(\Omega_k^n \Delta A_k^n) = 0$ and $\Omega_k^n \cap \Omega_{k'}^n = \emptyset$ for all k, $k' \in \mathbb{N}$ with $k \ne k'$.

For each $n, k \in \mathbb{N}$, fix an internal lifting with respect to the internal metric ${}^*\rho$ of the restriction of g_n to Ω_k^n; the lifting $\widetilde{g}_n^k$ is an internally measurable mapping from $(\Omega_k^n, \mathcal{A}_0 \cap \Omega_k^n)$ to $\{x \in {}^*X : \|x\| \le k\}$, where $\mathcal{A}_0 \cap \Omega_k^n$ is the internal algebra of $\mathcal{A}_0$-measurable subsets of Ω_k^n. Extend the double sequences Ω_k^n and $\widetilde{g}_n^k$ for $n, k \in \mathbb{N}$ to obtain internal sequences Ω_k^n and $\widetilde{g}_n^k$ with indices n, k now in ${}^*\mathbb{N}$. Also take the nonstandard extension $\left\{{}^*\varepsilon_{k\bar{j}} : k \in {}^*\mathbb{N}\right\}$ of the sequence $\left\{\varepsilon_{k\bar{j}} : k \in \mathbb{N}\right\}$. There is an unlimited $H^0 \in {}^*\mathbb{N}$ such that for any $n \le H^0$ and any $k, k' \le H^0$, (i) $\Omega_k^n \cap \Omega_{k'}^n = \emptyset$ when $k \ne k'$, (ii) $\widetilde{g}_n^k$ is an internally measurable mapping from $(\Omega_k^n, \mathcal{A}_0 \cap \Omega_k^n)$ to $\{x \in {}^*X : \|x\| \le k\}$, and (iii) $\bar{\mu}_0(\cup_{l=1}^k \Omega_l^n) \ge 1 - {}^*\varepsilon_{k\bar{j}} - \frac{1}{k}$. For each $n \in {}^*\mathbb{N}$ with $n \le H^0$, let $\Omega_0^n = \Omega \setminus \cup_{l=1}^{H^0} \Omega_l^n$; set $h_n(\omega) = 0 \in X$ for $\omega \in \Omega_0^n$, and set $h_n(\omega) = \widetilde{g}_n^l(\omega)$ for $\omega \in \Omega_l^n$, $1 \le l \le H^0$. Then, $\bar{\mu}(\|h_n\| \le k) \ge 1 - {}^*\varepsilon_{k\bar{j}} - \frac{1}{k}$, whence for $\bar{\mu}$-almost all $\omega \in \Omega$, $\|h_n(\omega)\|$ is limited in ${}^*\mathbb{R}^+$. For each $n \in \mathbb{N}$, h_n is an internal lifting of g_n, and $h_n : (\Omega, \mathcal{A}_0, \bar{\mu}_0) \to ({}^*X, {}^*\rho)$. The function h_n is also an internal lifting of g_n with respect to the measure μ_{0j} for each $j \in J$. Let E_n be a $\bar{\mu}$-null set in Ω such that ${}^*\rho(h_n(\omega), g_n(\omega)) \simeq 0$ for every $\omega \notin E_n$.

Let δ be the Prohorov metric on the space of Borel probability measures on (X, ρ). Fix any $j \in \bar{J}$. Since $\lim_{n\to\infty} \delta\left(\mu_j g_n^{-1}, \gamma_j\right) = 0$, the standard part of ${}^*\delta(\mu_{0j} h_n{}^{-1}, {}^*\gamma_j)$ goes to zero as n goes to infinity through standard values. Thus there is an unlimited $M_j \in {}^*\mathbb{N}$ such that for any unlimited $n \in {}^*\mathbb{N}$ with $n \le M_j$,

$$(6) \qquad {}^*\delta(\mu_{0j} h_n{}^{-1}, {}^*\gamma_j) \simeq 0.$$

Fix an unlimited element M^- of ${}^*\mathbb{N}$ such that $M^- \le M_j$ for every $j \in \bar{J}$. It follows from Lemma 5.1 that for any unlimited $n \in {}^*\mathbb{N}$ with $n \le M^-$, ${}^\circ h_n(\omega)$ exists for $\bar{\mu}$-almost all $\omega \in \Omega$ (and hence for μ_j-almost all $\omega \in \Omega$ for any given $j \in J$). Moreover, $\gamma_j = \mu_j({}^\circ h_n)^{-1}$ for every $j \in J$.

Let $\bar{X} = X \times [0, \infty)$ with the metric $\bar{\rho}$ defined as the summation on the two relevant metrics, and set $\bar{h}_n(\omega) = (h_n(\omega), \|h_n(\omega)\|)$ for $1 \le n \le H^0$. For any $l \in \mathbb{N}$, by applying Lemma 5.2 to the internal sequence $\bar{h}_n$, $1 \le n \le H^0$, and the standard compact set $K_l = \{x \in X : \|x\| \le l\} \times [0, l]$, we see that there is an unlimited $H_l \in {}^*\mathbb{N}$ and a $\bar{\mu}$-null set $S_l \subset \Omega$ such that for each unlimited $n \in {}^*\mathbb{N}$ with $n \le H_l$, $\bar{h}_n$ has the property that if $\bar{h}_n(\omega)$ has standard part in K_l and $\omega \notin S_l$, then for any standard $\varepsilon > 0$, there are infinitely many limited $k \in \mathbb{N}$ for which ${}^*\bar{\rho}(\bar{h}_k(\omega), \bar{h}_n(\omega)) < \varepsilon$. Fix an unlimited element $H \in {}^*\mathbb{N}$ such that $H \le H_l$ for each $l \in \mathbb{N}$, $H \le M^-$ and $H \le H^0$.

For each $\omega \in \Omega$, let $g(\omega)$ be the standard part in the space (X, ρ) of $h_H(\omega)$ when the standard part is defined, and set $g(\omega) = 0$ otherwise. For every $j \in J$, $\gamma_j = \mu_j g^{-1}$. Let S_0 be a $\bar{\mu}$-null set such that $\|h_H(\omega)\|$ is standardly finite for $\omega \notin S_0$. Let $S = (\cup_{l=0}^{\infty} S_l) \cup (\cup_{n=1}^{\infty} E_n)$; then $\bar{\mu}(S) = 0$. For any $\omega \notin S$, there is $l \in \mathbb{N}$ such that (1) the standard part with respect to the space $(\bar{X}, \bar{\rho})$ of $(h_H(\omega), \|h_H(\omega)\|)$ is in K_l, and (2) there is a strictly increasing sequence $\langle k_q \rangle$ in $\mathbb{N}$ such that the standard part of ${}^*\rho(h_{k_q}(\omega), h_H(\omega))$ goes to zero as q goes to infinity while $\|h_{k_q}(\omega)\| \leq l + 1$, which implies that $g_{k_q}(\omega)$ converges to $g(\omega)$ under the metric ρ within a ball of radius $l + 1$. Hence, for any $\omega \notin S$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^{\infty}$.

For each $y \in Y$, denote the functional $\langle \cdot, y \rangle$ on X by ϕ_y. The functional ϕ_y is continuous with respect to the weak* topology on X, but it may not be continuous with respect to the metric topology generated by ρ. Fix a closed subset F of $\mathbb{R}$, and for each $k \geq 1$, let $C_k = \phi_y^{-1}(F) \cap \{x \in X : \|x\| \leq k\}$. The set C_k is a weak* closed subset of the closed ball centered at 0 with radius k, and is therefore closed in (X, ρ).

For each $j \in \bar{J}$ and $n \in \mathbb{N}$, let ν_j^n denote the measure $\mu_j g_n^{-1}$. Fixing $j \in \bar{J}$, we know that the measures ν_j^n converges weakly to γ_j, so $\limsup_{n\to\infty} \nu_j^n(C_k) \leq \gamma_j(C_k)$. (See Theorem 6.1 of [**27**]). Using Equation (5), we see that for any given $y \in Y$, $\nu_j^n(\phi_y^{-1}(F)) \leq \nu_j^n(C_k) + \varepsilon_{kj}$, and hence

$$
\begin{aligned}
\limsup_{n\to\infty} \nu_j^n(\phi_y^{-1}(F)) &\leq \limsup_{n\to\infty} \nu_j^n(C_k) + \varepsilon_{kj} \\
&\leq \gamma_j(C_k) + \varepsilon_{kj} \leq \gamma_j(\phi_y^{-1}(F)) + \varepsilon_{kj}. \qquad (7)
\end{aligned}
$$

Therefore, $\limsup_{n\to\infty} \nu_j^n(\phi_y^{-1}(F)) \leq \gamma_j(\phi_y^{-1}(F))$, and so $\nu_j^n \phi_y^{-1}$ converges weakly to $\gamma_j \phi_y^{-1}$ on $\mathbb{R}$ (again by Theorem 6.1 of [**27**]). It follows that the sequence $\{\langle g_n, y\rangle\}_{n=1}^{\infty}$ of real-valued random variables converges in distribution to the random variable $\langle g, y\rangle$ on $(\Omega, \mathcal{A}, \mu_j)$.

To finish the proof of Case A, we note that since by assumption the sequence $\{g_n\}_{n=1}^{\infty}$ is uniformly μ_j-norm-integrable, for each $y \in Y$, the sequence $\{\langle g_n, y\rangle\}_{n=1}^{\infty}$ is also uniformly μ_j-integrable, whence $\lim_{n\to\infty} \int_\Omega \langle g_n, y\rangle d\mu_j = \int_\Omega \langle g, y\rangle d\mu_j$ (see [**6**, Theorem 5.4, p. 32]). This means that g is Gelfand μ_j-integrable, and the weak* limit of the Gelfand integrals $\int_\Omega g_n d\mu_j$ is $\int_\Omega g d\mu_j$.

To finish the proof of Case B, we fix $j \in J$, and an element y in the positive cone Y_+ of Y. We have shown that the sequence of $\{\langle g_n, y\rangle\}_{n=1}^{\infty}$ of nonnegative real-valued random variables converges in distribution to the random variable $\langle g, y\rangle$ on $(\Omega, \mathcal{A}, \mu_j)$. It now follows, as shown in [**6**, Theorem 5.3, p. 32], that

$$\int_\Omega \langle g(\omega), y\rangle d\mu_j \leq \liminf_{n\to\infty} \int_\Omega \langle g_n(\omega), y\rangle d\mu_j = \langle a_j, y\rangle. \tag{8}$$

Therefore, the non-negative function $\langle g, y\rangle$ is μ_j-integrable.

Since an arbitrary element z in Y can be expressed as the difference of two non-negative elements, it is thus clear that $\langle g, z\rangle$ is μ_j-integrable, and hence g is Gelfand μ_j-integrable. Moreover, for each $y \in Y_+$

$$\left\langle \int_\Omega g d\mu_j, y \right\rangle = \int_\Omega \langle g(\omega), y\rangle d\mu_j \leq \langle a_j, y\rangle,$$

so in terms of the ordering on X, $\int_\Omega g d\mu_j \leq a_j$.

To finish the proof, we again recall that for any $y \in Y$ and $j \in J$, the sequence $\{\langle g_n, y\rangle\}_{n=1}^\infty$ of real-valued random variables converges in distribution to the random variable $\langle g, y\rangle$ on $(\Omega, \mathcal{A}, \mu_j)$. If $\{\langle g_n, y\rangle\}_{n=1}^\infty$ is uniformly μ_j-integrable, then as shown in [**6**, Theorem 5.4, p. 32],

$$\lim_{n\to\infty} \int_\Omega \langle g_n, y\rangle d\mu_j = \int_\Omega \langle g, y\rangle d\mu_j = \langle a_j, y\rangle.$$

For any $j \in J$, if $\{g_n\}_{n=1}^\infty$ is uniformly μ_j-norm-integrable, then $\int_\Omega \langle g, y\rangle d\mu_j = \langle a_j, y\rangle$ for every $y \in Y$, and in this case $\int_\Omega g d\mu_j = a_j$. $\square$

6. Proof of Theorem 2.5

We close with the proof of Theorem 2.5, which is a generalization of Case B of Theorem 2.4; we often refer back to the proof of Theorem 2.4. First we show that if X satisfies the norm approximation property, then the sequence $\{g_n\}_{n=1}^\infty$ is weak* μ_j-tight. Fix $j \in J$. A proof similar to that given for Proposition 5.3 shows that for each $n \in \mathbb{N}$,

$$\int_\Omega \|g_n - f_n\|\, d\mu_j = \left\| \int_\Omega (g_n - f_n)\ d\mu_j \right\|,$$

and the Gelfand integrals for the sequence $\{g_n\}_{n=1}^\infty$ are uniformly norm-bounded. By assumption, the integrals $\int_\Omega \|f_n\|\, d\mu_j$ are uniformly bounded. It follows that the Gelfand integrals for the sequence $\{f_n\}_{n=1}^\infty$ are uniformly norm-bounded, so the same is true for the Gelfand integrals of the sequence $\{g_n - f_n\}_{n=1}^\infty$. Therefore, for some $M_j > 0$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\int_\Omega \|g_n\|\, d\mu_j &\leq \int_\Omega \|g_n - f_n\|\, d\mu_j + \int_\Omega \|f_n\|\, d\mu_j \\ &= \left\| \int_\Omega (g_n - f_n)\ d\mu_j \right\| + \int_\Omega \|f_n\|\, d\mu_j \leq M_j,\end{aligned}$$

and the result follows as before.

Next, we still denote $\bar{\mu}$ by $\mu_{\bar{j}}$, and we let $\bar{J} = J \cup \{\bar{j}\}$. We again use the metric ρ on X. It follows from our assumptions that the sequence $\{g_n\}_{n=1}^{\infty}$ is weak* μ_j-tight for each $j \in \bar{J}$. As before, for each $j \in \bar{J}$ the sequence $\varepsilon_{kj} = \sup_{n\geq 1} \mu_j(\{\omega \in \Omega : \|g_n(\omega)\| > k\})$ goes to zero as k goes to infinity.

Let ρ_2 be the metric on $X^2 = X \times X$ defined at (x_1, x_2), (x_1', x_2') by setting $\rho_2((x_1, x_2), (x_1', x_2')) = \max\{\rho(x_1, x_1'), \rho(x_2, x_2')\}$. Fix $j \in \bar{J}$. Since $\{f_n\}_{n=1}^{\infty}$ is uniformly μ_j-norm-integrable, it is weak* μ_j-tight. Therefore, the sequence $\{(g_n, f_n)\}_{n=1}^{\infty}$ of functions from $(\Omega, \mathcal{A}, \mu_j)$ to (X^2, ρ_2) is tight. For each $n \in \mathbb{N}$, let τ_j^n denote the Borel probability measure $\mu_j(g_n, f_n)^{-1}$ on (X^2, ρ_2); let ν_j^n and $(\nu')_j^n$, denote $\mu_j g_n^{-1}$ and $\mu_j {f_n}^{-1}$, respectively; these are the marginal measures of τ_j^n.

By the classical Prohorov Theorem, for each $j \in \bar{J}$, the tight sequence of measures $\langle \tau_j^n : n \in \mathbb{N} \rangle$ has a convergent subsequence in the topology of weak convergence of Borel measures on (X^2, ρ_2). As before, we may assume that $\{(g_n, f_n)\}_{n=1}^{\infty}$ is itself the sequence producing the measures τ_j^n that converge to Borel probability measures τ_j on (X^2, ρ_2). For each $j \in \bar{J}$, let γ_j and γ_j' be the respective marginal probability measures of τ_j on the first and second components of (X^2, ρ_2). The measures, ν_j^n and $(\nu')_j^n$ converge weakly to γ_j and γ_j', respectively.

Now as before, we define the $\mathcal{A}$-measurable partition $\{A_k^n, k \in \mathbb{N}\}$ of Ω, and we find a sequence of internal sets Ω_k^n, $k \in \mathbb{N}$, such that $\bar{\mu}(\Omega_k^n \Delta A_k^n) = 0$ and $\Omega_k^n \cap \Omega_{k'}^n = \emptyset$ for all $k \neq k'$. Also for each $n, k \in \mathbb{N}$, we find an internal lifting with respect to the internal metric ${}^*\rho$ of the restriction of g_n to Ω_k^n. The lifting $\widetilde{g}_n^k$ is an internal map from Ω_k^n to $\{x \in {}^*X : \|x\| \leq k\}$.

Extending the double sequences Ω_k^n and $\widetilde{g}_n^k$ for $n, k \in \mathbb{N}$ to internal sequences Ω_k^n and $\widetilde{g}_n^k$ for $n, k \in {}^*\mathbb{N}$, we define as before an unlimited $H^0 \in {}^*\mathbb{N}$ and internal functions $h_n : (\Omega, \mathcal{A}_0, \bar{\mu}_0) \to ({}^*X, {}^*\rho)$ for $n \leq H^0$ with $\|h_n(\omega)\|$ limited in ${}^*\mathbb{R}^+$ for $\bar{\mu}$-almost all $\omega \in \Omega$. For each $n \in \mathbb{N}$, h_n is an internal lifting of g_n with respect to the measure $\bar{\mu}$ (and μ_{0j} as well for each $j \in J$). Let E_n be a $\bar{\mu}$-null set in Ω such that ${}^*\rho(h_n(\omega), g_n(\omega)) \simeq 0$ for every $\omega \notin E_n$.

We may assume that for the same H^0 we have an internal sequence of functions $\{h_n'\}_{n=1}^{H^0}$ so that for each $n \in \mathbb{N}$, h_n' is an internal lifting from $(\Omega, \mathcal{A}_0, \bar{\mu}_0)$ to $({}^*X, {}^*\rho)$ for f_n and also an internal lifting of f_n with respect to the measure μ_{0j} for each $j \in J$.

Let δ be the Prohorov metric on the space of Borel probability measures on (X^2, ρ_2). Given $j \in \bar{J}$, the standard part of ${}^*\delta(\mu_{0j}(h_n, h_n')^{-1}, {}^*\tau_j)$ goes to zero as n goes to infinity through standard values. Thus there is an unlimited $M_j \in {}^*\mathbb{N}$ such that for each unlimited $n \in {}^*\mathbb{N}$ with

$n \le M_j$, ${}^*\delta(\mu_{0j}(h_n, h'_n)^{-1}, {}^*\tau_j) \simeq 0$. Fix an unlimited element M^- of ${}^*\mathbb{N}$ such that $M^- \le M_j$ for every $j \in \bar{J}$. It follows from Lemma 5.1 that for any unlimited $n \in {}^*\mathbb{N}$ with $n \le M^-$, ${}^\circ(h_n(\omega), h'_n(\omega))$ exists for $\bar{\mu}$-almost all $\omega \in \Omega$ (and hence for μ_j-almost all $\omega \in \Omega$ for any given $j \in J$). Moreover, $\tau_j = \mu_j({}^\circ(h_n, h'_n))^{-1}$ for every $j \in J$.

Let $\bar{X} = X \times [0, \infty)$ with the metric $\bar{\rho}$ defined as the summation on the two relevant metrics, and set $\bar{h}_n(\omega) = (h_n(\omega), \|h_n(\omega)\|)$ for $1 \le n \le H^0$. For any $l \in \mathbb{N}$, by applying Lemma 5.2 to the internal sequence $\bar{h}_n$, $1 \le n \le H^0$, and $K_l = \{x \in X : \|x\| \le l\} \times [0, l]$, we see that there is an unlimited $H_l \in {}^*\mathbb{N}$ and a $\bar{\mu}$-null set $S_l \subset \Omega$ such that for each unlimited $n \in {}^*\mathbb{N}$ with $n \le H_l$, $\bar{h}_n$ has the property that if $\bar{h}_n(\omega)$ has standard part in K_l and $\omega \notin S_l$, then for any standard $\varepsilon > 0$, there are infinitely many limited $k \in \mathbb{N}$ for which ${}^*\bar{\rho}(\bar{h}_k(\omega), \bar{h}_n(\omega)) < \varepsilon$. Fix an unlimited element $H \in {}^*\mathbb{N}$ such that $H \le H_l$ for any $l \in \mathbb{N}$, $H \le M^-$ and $H \le H^0$.

For each $\omega \in \Omega$, let $(g(\omega), f(\omega))$ be the standard part in the space (X^2, ρ_2) of $(h_H(\omega), h'_H(\omega))$ when the standard part is defined, and set $(g(\omega), f(\omega)) = (0, 0)$ otherwise. For any $j \in J$, $\tau_j = \mu_j(g, f)^{-1}$, and in particular, $\gamma_j = \mu_j g^{-1}$ and $\gamma'_j = \mu_j f^{-1}$. Let S_0 be a $\bar{\mu}$-null set such that $\|h_H(\omega)\|$ is standardly finite for $\omega \notin S_0$. Let $S = (\cup_{l=0}^{\infty} S_l) \cup (\cup_{n=1}^{\infty} E_n)$; then $\bar{\mu}(S) = 0$. For any $\omega \notin S$, there is $l \in \mathbb{N}$ such that (1) the standard part with respect to the space $(\bar{X}, \bar{\rho})$ of $(h_H(\omega), \|h_H(\omega)\|)$ is in K_l, and (2) there is a strictly increasing sequence $\langle k_q \rangle$ in $\mathbb{N}$ such that the standard part of ${}^*\rho(h_{k_q}(\omega), h_H(\omega))$ goes to zero as q goes to infinity while $\|h_{k_q}(\omega)\| \le l + 1$, which implies that $g_{k_q}(\omega)$ converges to $g(\omega)$ under the metric ρ within a ball of radius $l + 1$. Hence, for any $\omega \notin S$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^{\infty}$.

For each $y \in Y$, denote the functional $\langle \cdot, y \rangle$ on X by ϕ_y. The same proof as before shows that $\nu_j^n \phi_y^{-1}$ converges weakly to $\gamma_j \phi_y^{-1}$. A similar proof shows that $(\nu')_j^n \phi_y^{-1}$ converges weakly to $\gamma'_j \phi_y^{-1}$.

Fix an element y in Y. On X^2, let $\psi_y(x_1, x_2) = \langle x_1 - x_2, y \rangle$. Take any closed subset F of $\mathbb{R}$. For each $k \ge 1$, let $B_k = \{(x_1, x_2) \in X^2 : \|x_1\| \le k, \|x_2\| \le k\}$. Set $D_k = \psi_y^{-1}(F) \cap B_k$, and let ψ_y^k be the restriction of ψ_y to B_k. It is obvious that $D_k = (\psi_y^k)^{-1}(F)$. Since ψ_y^k is continuous on (B_k, ρ_2), $(\psi_y^k)^{-1}(F)$ is a closed subset of B_k, and hence a close set in (X^2, ρ_2). With arguments similar to those given before, it follows that $\tau_j^n \psi_y^{-1}$ converges weakly to $\tau_j \psi_y^{-1}$ as n goes to infinity.

Next we note that the proof given for the g_n's in finishing the proof of Case A of Theorem 2.4 also shows, when applied to the f_n's, that f is Gelfand μ_j-integrable, and the weak* limit of the Gelfand integrals $\int_\Omega f_n d\mu_j$ is $\int_\Omega f d\mu_j$.

Fix $j \in J$, and an element y in the positive cone Y_+ of Y. It is shown above that the sequence $\{\psi_y(g_n, f_n)\}_{n=1}^{\infty}$ of real-valued random variables converges in distribution to the random variable $\psi_y(g, f)$ on $(\Omega, \mathcal{A}, \mu_j)$. Since $\psi_y(g_n, f_n)(\omega) = \langle g_n(\omega) - f_n(\omega), y\rangle \geq 0$ for any $n \in \mathbb{N}$, $\omega \in \Omega$, $\psi_y(g, f)$ is also a non-negative random variable. It now follows, as shown in [**6**, Theorem 5.3, p. 32], that

$$\begin{aligned} \int_\Omega \langle g - f, y\rangle d\mu_j &= \int_\Omega \psi_y(g, f) d\mu_j \\ &\leq \liminf_{n\to\infty} \int_\Omega \psi_y(g_n, f_n) d\mu_j \\ &= \liminf_{n\to\infty} \left[\int_\Omega \langle g_n, y\rangle d\mu_j - \int_\Omega \langle f_n, y\rangle d\mu_j \right] \\ &= \langle a_j, y\rangle - \int_\Omega \langle f, y\rangle d\mu_j. \end{aligned} \tag{9}$$

Therefore, the non-negative function $\langle g - f, y\rangle$ is μ_j-integrable. Since $\langle f, y\rangle$ is μ_j-integrable, we know that $\langle g, y\rangle$ is μ_j-integrable.

Since an arbitrary element z in Y can be expressed as the difference of two non-negative elements, it is thus clear that $\langle g, z\rangle$ is μ_j-integrable, and hence g is Gelfand μ_j-integrable. By Equation (9), we have for $y \in Y_+$

$$\langle \int_\Omega g d\mu_j, y\rangle - \langle \int_\Omega f d\mu_j, y\rangle \leq \langle a_j, y\rangle - \langle \int_\Omega f d\mu_j, y\rangle, \tag{10}$$

which implies that $\langle \int_\Omega g d\mu_j, y\rangle \leq \langle a_j, y\rangle$. Hence $\int_\Omega g d\mu_j \leq a_j$. The proof of the two statements of equality is the same as before. □

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