Illinois Journal of Mathematics
Volume 54, Number 4, Winter 2010, Pages 1451–1461
S 0019-2082

REPRESENTING MEASURES IN POTENTIAL THEORY AND AN IDEAL BOUNDARY

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For my friend Donald Burkholder, a great mathematician

ABSTRACT. A primary motivating application of the author's work on measure theory was a nonstandard construction of standard representing measures for positive harmonic functions. That work yielded new standard weak convergence methods for constructing such measures on spaces of extreme harmonic functions in very general settings. The search for a Martin-type ideal boundary for the placement of those measures resulted in a new almost everywhere regular boundary that supported the representing measures for a large proper subclass of all nonnegative harmonic functions. In this note, we outline the construction of the rich measure spaces that are now called Loeb measure spaces in the literature. We then review the application of these measure spaces to the construction of representing measures. We finish with the problem of constructing an appropriate boundary associated with the nonstandard construction of general representing measures that supports all of those measures.

1. Introduction

In 1975 [10], the author constructed a class of standard measure spaces formed on nonstandard models. These spaces, now called "Loeb spaces" in the literature, are very close to underlying "internal" spaces and share their combinatorial properties. A motivating example, developed at the same time as the basic measure theory, was the construction of representing measures in potential theory and applications to ideal boundaries [12]. In this article, we review the construction of representing measures and the resulting, weak-limit

Received June 9, 2010; received in final form November 15, 2010.
2010 *Mathematics Subject Classification.* 03H10, 28E05, 31C35.

result valid for quite general potential theories. We then discuss the author's ongoing search for a Martin-type boundary appropriate for that construction. We begin with a brief survey of basic nonstandard analysis and "Loeb" measure theory; the survey is an invitation to read and not a substitute for the many articles and books published over the years on these subjects. See [16] for more extensive background.

2. Basic nonstandard analysis

One thinks of a standard mathematical model as a world that exists in some sense. For example, we think of the real numbers as having an existence independent of what we may know about them. Theorems in an appropriate formal language form correct statements about such a model. It is important to recognize the distinction between the names of objects in a standard model along with statements using such names in a formal language, and the objects themselves. For example, the number five has many names such as 5 in base ten, 101 in binary, V in Roman numerals. The reason to emphasize this distinction is that for each standard mathematical model there are other mathematical objects, called nonstandard models, for which all the names and theorems for the standard model have a meaning and are correct for each nonstandard model. Informally, if we fix a nonstandard model, what we have are two worlds, the standard and the nonstandard, and the theorems about the first are also correct statements about the second. The foundation for the application of this fact to analysis, called nonstandard analysis, is due to Abraham Robinson [18]. His nonstandard models for the real number system contain infinitely large and infinitely small positive numbers together with all of the numbers in the original real number system.

One way to explain Robinson's result is to invoke a theorem of Kurt Gödel. Take a name not use for anything in the standard number system—for example, Bach. To the theorems about the standard real number system add new statements: "Bach is bigger than 1," "Bach is bigger than 2," etc. Add one such statement for each natural number. The standard number system is not a model for the collection of theorems augmented by these statements about Bach. There is no number simultaneously bigger than $1, 2, 3$, etc. The standard number system is, however, a model for any finite subset of the augmented collection of statements. To see this, fix a finite subset of the augmented collection. Find the biggest number named in these statements, and let Bach be the name of a number that is even bigger. Since every finite subset of our augmented collection of statements has a model, it follows from a result of Gödel that the entire augmented collection of statements has a model. That is there is a number system for which all the theorems about the real numbers hold, but there is a number in that system, call it Bach, that is bigger than $1, 2, 3$, etc. Bach's reciprocal, 1 divided by Bach, is then a positive infinitesimal number.

Another approach to understanding Robinson's result is to construct a simple number system with infinitesimals using sequences of real numbers and a free ultrafilter $\mathcal{U}$ on the natural numbers. Two sequences represent the same nonstandard number if they agree on a set $U \in \mathcal{U}$. A constant sequence represents an ordinary number. For example, the sequence $5, 5, 5, \ldots,$ represents the number 5. On the other hand, a sequence of positive real numbers tending to zero such as the sequence $1, 1/2, 1/3, 1/4, \ldots,$ becomes a positive infinitesimal. A sequence of real numbers increasing to infinity such as the sequence $1, 2, 3, \ldots,$ becomes an infinitely large number.

Rather than thinking of constructions, however, it is better in practice to work with the properties of the extended number system. In doing so, one should keep in mind the fact that any theorem for the ordinary numbers is also a theorem, when properly interpreted, for the enlarged number system.

What is meant by saying "when properly interpreted"? Briefly, when we say "all" subsets of a given set, we can't formally specify what we mean. Even for the set of natural numbers, the idea of all subsets cannot be formalized. Ordinary language, for example, can only describe at most countably many subsets of the natural numbers. This inability to formalize the notion of "all subsets" means that when interpreting theorems in the nonstandard model, we can cheat. We don't interpret the word "all" to really mean "all." We work instead with what are called *internal* sets, and interpret "all sets" to mean all internal sets.

If A is a set in the standard model, then ${}^*A$, called the nonstandard extension of A, is the set in the nonstandard model with the same name and formal properties as A. Nonstandard extensions of standard sets and elements of nonstandard extensions of standard sets are internal sets. Any object that can be described using only the names of known internal objects is also internal. An object that is not internal is called *external*.

Important for applications is the fact that the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ has been extended along with the real numbers. We use ${}^*\mathbb{N}$ to denote the extended system of natural numbers. The theorem that says every positive real number is within distance one of a natural number is still valid for the extended number system. Since one can list the predecessors in $\mathbb{N}$ of any standard natural number, the new elements of ${}^*\mathbb{N}$ are all greater than any given standard natural number. We use ${}^*\mathbb{N}_\infty$ to denote the set of new members of ${}^*\mathbb{N}$, all of which are larger than any ordinary natural number. The set ${}^*\mathbb{N}_\infty$ is clearly external, since every nonempty, internal subset of ${}^*\mathbb{N}$ must have a first element. If ${}^*\mathbb{N}_\infty$ were internal, then subtracting 1 from a first element of ${}^*\mathbb{N}_\infty$ would yield the last ordinary natural number. Since ${}^*\mathbb{N}_\infty$ is external, it follows that the set of ordinary natural numbers is also external since ${}^*\mathbb{N}_\infty$ can be described as the complement of that set in ${}^*\mathbb{N}$.

It is also important to know that if ρ is an element of the extended real number system, which we denote by ${}^*\mathbb{R}$, and $|\rho| \leq n$ for some ordinary $n \in \mathbb{N}$, then there is a unique real-number r such that $\rho = r + \varepsilon$ where ε is infinitesimal, that is, $\forall n \in \mathbb{N}$, $|\varepsilon| < 1/n$. The real number r is the supremum in $\mathbb{R}$ of all ordinary real numbers smaller or equal to ρ. We call r the standard part of ρ and write $r = \mathrm{st}(\rho)$. In general if $\rho \in^* \mathbb{R}$ and $|\rho| \leq n$ for some ordinary $n \in \mathbb{N}$, we will say ρ is *limited*; otherwise, we will say ρ is *unlimited.*

To apply these facts to calculus, consider the problem of finding the area above the x-axis and under the graph of the curve $y = x^2$ between 0 and 1. If we choose a natural number n and divide the interval between 0 and 1 into n intervals of length $1/n$, then the sum $(1/n)^2 \cdot 1/n + (2/n)^2 \cdot 1/n + \cdots + 1^2 * 1/n$ is an approximation to the area under the curve; the larger the value of n, the better the approximation. If we take an unlimited natural number n, we have a sum that equals the area except for an infinitesimal error. The area under the curve is the standard part of that sum.

Here however, we come to the question "what is meant by the sum?" First let's consider the question "what is meant by a finite set?" In the standard world, a set is finite if it can be enumerated with natural numbers finishing with a largest natural number. Having extended the real numbers and the natural numbers we have infinitely large, that is, unlimited natural numbers. If there is an internal bijection from a set in the nonstandard model onto an initial segment of ${}^*\mathbb{N}$ ending with an unlimited natural number, then the set is called a hyperfinite set. Hyperfinite sets are infinite sets, but they have all of the formal properties of finite sets. In particular, since we can sum any finite set of real numbers, the summing function in the nonstandard model also gives an answer for any hyperfinite subset of ${}^*\mathbb{R}$.

Hyperfinite sets play a central role in the applications of nonstandard analysis, to many areas of mathematics beyond the calculus. In probability, for example, it is easy to analyze a finite coin toss. It is harder to analyze an infinite coin toss. Any particular outcome has zero probability. On the other hand, in our nonstandard real number system there are unlimited natural numbers. We can choose such a number, again call it Bach, and, at least in our imagination, we can toss the coin Bach times. Now any particular outcome has probability equal to one divided by two raised to the power Bach. Moreover, this hyperfinite coin tossing space contains all standard infinite coin tosses.

We will describe in the next section the author's construction in [10] making this nonstandard experiment and its generalizations, with probabilities given by nonstandard real numbers, into ordinary probability spaces with real probabilities. These pairs of spaces, one with nonstandard probabilities and the other with ordinary probabilities, are examples of what are called "Loeb spaces" in the literature. One can parametrize ordinary probability experiments with these spaces. The underlying space of points is a set in a nonstandard model; the nonstandard space and the corresponding standard probability space are very close.

As shown by Anderson [1], hyperfinite coin tosses can be used to form a model for Brownian motion. One divides time up into infinitesimal intervals, and at the beginning of each interval tosses a coin. If the toss is a head, one moves to the right; if the toss is a tail, one moves to the left. The step size is the square root of the time change. These random walks, one for each hyperfinite coin tossing sequence, form a good underlying probability space for Brownian motion. Anderson also showed that the corresponding Loeb space generates Wiener measure.

Anderson's construction of Brownian motion can be viewed as a nonstandard formulation of Donsker's theorem. (See, for example, [3].) Other nonstandard treatments of weak convergence of measures can be found in [2] and [13]. We will discuss in a later section, the author's use of nonstandard analysis to produce a new, standard, weak-limit procedure for obtaining representing measures for positive harmonic functions in rather general settings.

Hyperfinite sets and the corresponding Loeb spaces play an important role in mathematical economics. A central problem in that subject is to study equilibria in economies with a vary large number of individuals when each individual has only a negligible influence on the economy. For this, it is quite natural to consider an economy with a hyperfinite number of individuals, each individual having only an infinitesimal influence on the economy.

A fundamental application of Loeb spaces to probability theory and to mathematical economics is Yeneng Sun's work constructing an appropriate space to represent a continuum of independent random variables in probability theory or independent agents in an economy. As Doob indicated in 1937, [6], whatever way one approaches this problem, the usual measure-theoretic tools fail. Sun has shown in Proposition 7.33 of [16] that no matter what kind of measure spaces, even Loeb measure spaces, one might take as the parameter space and sample space of a process, independence and joint measurability with respect to the classical measure-theoretic product, i.e., formed using measurable rectangles as in [19], are never compatible with each other except for trivial cases. In [21], [22], [23], and Chapter 7 of [16], Sun has shown that a construction overcoming these measure-theoretic problems is obtained by forming the internal product of internal factors, and then taking not just the Loeb space of each factor, but also the Loeb space of the internal product.

3. Nonstandard measure theory

We now briefly outline the construction of Loeb measure spaces. The principal device used is $\aleph_1$-saturation. This means that any ordinary sequence taken from an internal set is the beginning of an internal sequence, using ${}^*\mathbb{N}$, from that set. We will restrict our discussion to spaces formed using hyperfinite sets. Working in an $\aleph_1$-saturated nonstandard model, we can construct a hyperfinite set Λ as the set of elementary outcomes in a conceptual experiment in the "nonstandard world." For coin tossing, for example, Λ can be the set of internal sequences of -1's and 1's of length $\eta \in^* \mathbb{N}_\infty$. Given such a hyperfinite Λ, we will let $\mathcal{C}$ consist of all internal subsets of Λ. The collection $\mathcal{C}$ is an internal σ-algebra, but it is also an algebra in the ordinary sense. Suppose P is an internal probability measure on $(\Lambda, \mathcal{C})$. For the coin tossing experiment, for example, each internal set A with internal cardinality $|A|$, would be given the probability $P(A) = |A|/2^\eta$ in ${}^*[0,1]$. For a general internal probability measure P on $(\Lambda, \mathcal{C})$, we can form a finitely additive real-valued measure $\widehat{P}$ on $(\Lambda, \mathcal{C})$ with values in the real interval $[0,1]$ by setting $\widehat{P}(A) = \mathrm{st}(P(A))$. The question is, "Can we extend $\widehat{P}$ to a countably additive measure on $\sigma(\mathcal{C})$, that is, the σ-algebra generated by $\mathcal{C}$?" The answer is "Yes we can." We can extend $\widehat{P}$ to a measure μ defined on the measure completion $L_\mu(\mathcal{C})$ of $\sigma(\mathcal{C})$, and thus obtain a standard measure space $(\Lambda, L_\mu(\mathcal{C}), \mu)$ on Λ, by using the standard Carathéodory Extension Theorem.

To apply this technique, we note that when a sequence $\langle A_i : i \in \mathbb{N}\rangle$, indexed by the ordinary natural numbers, consists of pairwise disjoint elements of $\mathcal{C}$ and the union A is also in $\mathcal{C}$ then A is actually a finite union since all but a finite number of the A_i's are empty. Here is the simple proof: Using $\aleph_1$-saturation, we extend the sequence $\langle A_i : i \in \mathbb{N}\rangle$ to an internal sequence $\langle A_i : i \in^* \mathbb{N}\rangle$; the set

$$\left\{ m \in {}^*\mathbb{N} : A \subseteq \bigcup_{1 \le i \le m} A_i \right\}$$

is internal and contains ${}^*\mathbb{N}_\infty$, so it must contain some standard natural number since ${}^*\mathbb{N}_\infty$ is external. It now follows that $\widehat{P}(A) = \sum_{i \in N} \widehat{P}(A_i)$. By the Carathéodory Extension Theorem, the finitely additive measure $\widehat{P}$ has a unique σ-additive extension μ defined on the completion $L_\mu(\mathcal{C})$ of the σ-algebra $\sigma(\mathcal{C})$.

When used in probability theory, the above general construction allows one to tackle problems of continuous parameter stochastic processes using the combinatorial tools available for discrete parameter processes. Examples are the author's construction of Poisson processes in [10] and Anderson's representation of Brownian Motion and the Itô integral in [1]. As demonstrated by the work of Keisler [8], Fajardo and Keisler, Sun, and others, these measure spaces and the non-hyperfinite generalizations have special closure properties not shared by even Lebesgue measure spaces. Their use has yielded new standard-analysis results by many researchers in areas such as probability theory, potential theory, mathematical economics and mathematical physics. As noted in [15], they form the prototype for a class of rich measure spaces using a technique originating in [7].

4. Representing measures in potential theory

The first application after coin tossing of the measure theory described above was a construction of representing measures for nonnegative harmonic functions; that construction works for very general potential theories (see [11]). To illuminate the construction, we start with harmonic functions on the unit disk in the complex plane $\mathbb{C}$. Let D_r denote the open disk $\{z \in \mathbb{C} : |z| < r\}$, and let $D = D_1$. Let C_r be the circle $\{z \in \mathbb{C} : |z| = r\}$, and let $C = C_1$. All measures we consider will be Borel measures. Let $\mathbf{P}(z, x)$ be the Poisson Kernel $(|z|^2 - |x|^2)/|z - x|^2$, and let x_0 denote the origin. We use $\mathcal{H}^1$ to denote the set of all positive harmonic functions on D taking the value 1 at x_0. The set $\mathcal{H}^1$ is convex and compact with respect to the topology of uniform convergence on compact subsets of D, that is, the ucc topology.

It is well known that every continuous function on C has a harmonic extension on D and that not every harmonic function on D is obtained in this way. On the other hand, by the Riesz–Herglotz Theorem there is for each $h \in \mathcal{H}^1$ a probability measure ν_h on C such that

$$h = \int_C \mathbf{P}(z, \cdot)\nu_h(dz).$$

The mapping $z \mapsto \mathbf{P}(z, \cdot)$ from C into $\mathcal{H}^1$ (with the ucc topology) is a homeomorphism. We may think of ν_h as a measure either on C or on the collection of harmonic functions $\{\mathbf{P}(z, \cdot) : z \in C\}$. The latter point of view is that of Martin boundary theory (see [5]) and Choquet theory. The simplest realization of Choquet theory deals with a triangle. Each point inside and on a triangle is represented by a unique affine weight on the extreme points of the triangle, i.e., on the vertices. For the compact, convex set $\mathcal{H}^1$, the extreme points are the functions $\{\mathbf{P}(z, \cdot) : z \in C\}$, and each $h \in \mathcal{H}^1$ is represented by a unique probability measure ν_h on this set. While the usual construction of ν_h is simple for the disk, it does not generalize without going to an ideal boundary. The measure theory discussed above does yield a generalizable construction of ν_h by extracting a measure from the function h that would otherwise be lost on general domains. We give a brief description of that construction. More details can be found in the original work [11], [12].

First, we recall that for each circle C_r and each point $x \in D_r$, there is by the Riesz Representation Theorem a Borel measure μ_r^x, called harmonic measure for x and r, that gives the value at x of the harmonic extension of any continuous function on C_r. Moreover, the uniform probability measure on C_r obtained from Lebesgue measure is the harmonic measure $\mu_{x_0}^r$ with respect to the origin x_0. Given $h \in \mathcal{H}^1$, the measures $h \cdot \mu_{x_0}^r$, $0 < r < 1$, are probability measures, and ν_h is the weak* limit as the radius r tends to 1. This construction of ν_h does not work for more general domains and potential theories, but the following modification is valid in these more general settings.

Fix $h \in \mathcal{H}^1$. For each $r < 1$, let $\{A_i^r\}$ form an interval partition of C_r, and choose $y_i^r \in A_i^r$. Let $\delta_{y_i^r}$ denote unit mass at the point y_i^r. The net of measures $\sum_i h(y_i^r)\mu_{x_0}^r(A_i^r) \cdot \delta_{y_i^r}$ converges in the weak* topology to the measure ν_h on C. The direction for this net is given by letting r tend to 1 and refining the partitions $\{A_i^r\}$. To see that ν_h is in fact the weak* limit of this net, note that the integral of any continuous function f with respect to one of these measures with support in C_r is a Riemann sum approximation to the integral of f with respect to the measure $h \cdot \mu_{x_0}^r$.

Instead of a finite combination of measures concentrated on the points of D, we want a combination of point masses on the function space $[0, +\infty]^D$. Given r and a partition $\{A_i^r\}$ of C_r, the function $x \mapsto \mu_x^r(A_i^r)$ is a harmonic function on D_r. It is the solution of the Dirichlet problem for the function that is 1 on A_i^r and 0 on the rest of C_r. When we divide by $\mu_{x_0}^r(A_i^r)$, the new function is equal to 1 at the origin x_0. Let δ_i^r be unit mass on the function that is equal to $\mu_x^r(A_i^r)/\mu_{x_0}^r(A_i^r)$ in D_r and is identically 0 on and outside C_r; the point mass δ_i^r is a measure on the function space $[0, +\infty]^D$ supplied with the product topology. By equicontinuity (see [9]), the restriction of the product topology is the ucc topology on the set of positive harmonic functions on D_r taking the value 1 at x_0. A nonstandard proof, given next, shows that ν_h is the weak* limit as r approaches 1 and the partitions $\{A_i^r\}$ are refined. The limit measure is supported by the set

$$\big\{\mathbf{P}(z, \cdot) : z \in C\big\} \subset \mathcal{H}^1 \subset [0, +\infty]^D.$$

This weak limit construction of ν_h in [12] was new and extends to rather general elliptic and parabolic differential equations on a locally compact, but not compact, connected and locally connected domain W (see [11], [12], [14]). It does not use the Martin boundary. For the generalization, one replaces the disks D_r with an increasing sequence of relatively compact, Dirichlet regular, domains $W_i \subset W$ so that for each compact set $K \subset W$, there is an i with $K \subset W_i$. For elliptic differential equations, a single point x_0 plays the part of the origin. The net of measures $\sum_i h(y_i^r)\mu_{x_0}^r(A_i^r) \cdot \delta_i^r$ have ν_h as a weak* limit as r approaches 1 and the partitions $\{A_i^r\}$ of ∂W_i are refined.

The proof in [12] that this construction works uses results from [2] and [13] to interpret the construction of ν_h in [11] as a weak* limit. Here, specialized to the case of the disk D, is that construction of ν_h from [11] and its subsequent interpretation as a weak* limit.

We start with a circle $C_r \subset^* D$ with $r \simeq 1$, and an interval partition $\{A_i^r\}$ so fine that every standard harmonic function has infinitesimal variation on each set A_i^r. Suppressing the superscript r, we have

$$\forall x \in D \quad h(x) = \int_{C_r}^{*} h(y)\, d\mu_x(y) \simeq \sum_i^{*} h(y_i)\mu_{x_0}(A_i)\frac{\mu_x(A_i)}{\mu_{x_0}(A_i)}.$$

The family of weights ${}^*h(y_i)\mu_{x_0}(A_i)$ is made into an ordinary probability measure μ_h using the general measure theory described above. The measure μ_h is supported by the set of nonstandard harmonic functions $\mu_x(A_i)/\mu_{x_0}(A_i)$. This is an internal set of positive, internal harmonic functions on D_r, with each function taking the value 1 at x_0. The mapping S on this set of functions given by the formula

$$S(g)(x) = {}^\circ\big(g(x)\big) \quad \forall x \in D$$

is the standard part mapping with respect to $\mathcal{H}^1$ supplied with the ucc topology. The measurability of S, established for this special case in [11], allows one to project the measure μ_h onto $\mathcal{H}^1$. The process preserves affine combinations of harmonic functions and yields representing measures, so by a corollary of a result by Cartier, Fell, and Meyer (see [11]), the final measure is the unique representing measure ν_h on the extreme points $\{\mathbf{P}(z,\cdot) : z \in C\}$ of $\mathcal{H}^1$. In [12], the projection of the measure μ_h using S is interpreted as taking the weak* limit of the standard net of measures described above.

There remains the problem of finding a Martin type boundary associated in a natural way with the above construction for a general domain W. What is needed is a boundary that will support all representing measures, just as C supports representing measures for the unit disk. An earlier effort in [11], later joined by Jürgen Bliedtner in [4], resulted in a rich, almost everywhere regular boundary associated with the space of uniform limits of positive, bounded harmonic functions. These are called "sturdy harmonic functions" in [4]. The boundary in [4] does not, however, support all representing measures.

5. New effort to form ∂W

Generalizing the setup outlined above for the unit disk D, we start, as already noted, with a locally compact, noncompact, connected and locally connected domain W and its nonstandard extension ${}^*W$. We fix an internally Dirichlet regular and relatively compact domain U in ${}^*W$ containing the extension of every standard compact subset of W. We choose a standard point $x_0 \in W$ that plays the part played by the origin in D.

Fix a partition $\{A_i\}$ of ∂U analogous to the partition of the circle C_r for $r \simeq 1$. That is, the variation of the extension of each standard, positive harmonic function on W has infinitesimal variation A_i. We now have an internal class of functions $\{\frac{\mu_\cdot(A_i)}{\mu_{x_0}(A_i)}\}$ each extended with 0 outside U. This class projects to a family $\mathcal{H}_P(W)$ of standard functions $P(\frac{\mu_\cdot(A_i)}{\mu_{x_0}(A_i)}) \subset \mathcal{H}^1(W)$. For each standard point x in W, $P(\frac{\mu_\cdot(A_i)}{\mu_{x_0}(A_i)})(x) = \mathrm{st}(\frac{\mu_x(A_i)}{\mu_{x_0}(A_i)})$. The map P is the standard part map with respect to the product topology, which is the ucc topology when restricted to $\mathcal{H}^1(W)$. Since $\mathcal{H}_P(W)$ is the image of an internal set, it follows from a result of W. A. J. Luxemburg [17] that $\mathcal{H}_P(W)$ is a compact subset of $\mathcal{H}^1(W)$ supplied with the ucc topology. As for the case of the unit disc, $\mathcal{H}_P(W)$ contains the extreme elements of $\mathcal{H}^1(W)$ supporting representing measure ν_h for every $h \in \mathcal{H}^1(W)$. The problem is to Choose U so that $\mathcal{H}_P(W)$ may be attached to W as at least part of a compactifying boundary.

A simple case starts with the assumption that W already has a nice compactifying boundary, ∂W, with nonstandard extension ${}^*\partial W$ contractible to ∂U so that if A_i and A_j on ∂U are in the monad of the same $z \in \partial W$ then $P(\frac{\mu.(A_i)}{\mu_{x_0}(A_i)}) = P(\frac{\mu.(A_j)}{\mu_{x_0}(A_j)})$. By the Permanence Principle, the map from ∂W to $\mathcal{H}_P(W)$ with the product topology is then continuous.

A counter example to this assumption is presented by a region W in 3-space between a sphere of radius 1 and a larger sphere of radius 2 with the two spheres touching only at one point z_0. The point z_0 is replaced by an infinite number of extreme elements of $\mathcal{H}^1(W)$ in the Martin compactification of W.

A principal weapon in the search for an appropriate compactification of W uses the following fact noted by S. Salbany and T. Todorov in [20]: The space ${}^*W$ is compact when supplied with the S-topology. The S-topology is the topology on ${}^*W$ generated by the nonstandard extensions of standard open sets. Compactifications of W are obtained by taking quotients of this space. Since W is locally compact, equivalence classes of near-standard points, i.e., points in the monads of standard points, form a homeomorphic image of W. It is the equivalence classes of non-near-standard points that form the points of the boundary, ∂W. The problem remains in the author's ongoing research to find the right equivalence relation for the non-near-standard points.

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