Hilbert schemes of points

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Outline

Hilbert scheme of points on $\mathbb{C}^2$

Hilbert scheme of points on a Deligne-Mumford stack

Hilbert scheme and q, t-Catalan numbers

Hilbert scheme of points on $\mathbb{C}^2$

$\mathrm{Hilb}^n(\mathbb{C}^2)$ is a scheme that parameterizes 0-dimensional subschemes $Z \subset \mathbb{C}^2$ satisfying $\dim \mathcal{O}_Z = n$.

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Example

- $n = 1$. $\mathrm{Hilb}^1(\mathbb{C}^2) = \mathbb{C}^2$.
- $n = 2$. $\mathrm{Hilb}^2(\mathbb{C}^2) = \left(\mathrm{Bl}_{\mathcal{I}}\mathbb{C}^4\right)/S_2$, where

$$\mathcal{I} = (x_1 - x_2, y_1 - y_2).$$

- For general n. $\mathrm{Hilb}^n(\mathbb{C}^2) = \left(\mathrm{Bl}_{\mathcal{I}}\mathbb{C}^{2n}\right)/S_n$, where

$$\mathcal{I} = \cap_{1\le i<j\le n}(x_i - x_j, y_i - y_j).$$

Properties of $\mathrm{Hilb}^n(\mathbb{C}^2)$:

- $\mathrm{Hilb}^n(\mathbb{C}^2)$ is smooth and connected.
- $\mathrm{Hilb}^n(\mathbb{C}^2)$ has a cellular decomposition.
- There is a Hilbert-Symm morphism $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$.
- $\mathrm{Hilb}^n(\mathbb{C}^2)$ is holomorphic symplectic, hence gives a crepant resolution of $\mathrm{Sym}^n(\mathbb{C}^2)$.

Hilbert scheme of points on a Deligne-Mumford stack (a project suggested by J. Starr)

Assume $k = \bar{k}$,
$\mathscr{X}$ is a tame DM stack / k and is a global quotient,
the coarse moduli space X is (quasi-)projective.

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Definition

$\mathrm{Hilb}^n(\mathscr{X})$ *is the (quasi-)projective scheme that represents the functor*

$$(k\text{-}Schemes) \to (Sets)$$

$$T \to \left\{ \mathscr{C} \subset \mathscr{X} \times T \;\middle|\; \begin{array}{l} \mathscr{C} \textit{ is a closed substack,} \\ \textit{finitely presented, flat and proper}/T, \\ \textit{satsify the Hilbert polynomial condition (*)} \end{array} \right\}$$

Recall: $\forall$ coherent $\mathcal{O}_{\mathscr{X}}$-module $\mathcal{F}$, define

$$\chi(\mathscr{X},\mathcal{F}) = \sum_{i=0}^{\infty}(-1)^i \dim_k H^i_{\text{ét}}(\mathscr{X},\mathcal{F}).$$

The Hilbert polynomial of $\mathcal{F}$, $P_{\mathcal{F}} : K^0(\mathscr{X}) \to \mathbb{Z}$, is defined as

$$[\mathcal{E}] \to \chi(\mathscr{X},\mathcal{E}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathcal{F}), \quad \forall \text{ locally free sheaf } \mathcal{E}.$$

Condition (*): $P_{\mathcal{O}_{\mathscr{C}_t}}(\mathcal{E}) = n\,\mathrm{rank}\mathcal{E} \quad \forall t \in T$.

Theorem

Let $\mathscr{X}$ be a smooth 2-dim tame DM stack with (quasi-)projective coarse moduli space X.
Then $\mathrm{Hilb}^n(\mathscr{X})$ is smooth and (quasi-)projective for all $n \in \mathbb{N}$.

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Idea: $\exists$ étale covering $\{X_i \to X\}$, scheme U_i with G_i-action,

$$\begin{array}{ccc} [U_i/G_i] & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{\text{étale}} & X \end{array}$$

Let $W =$ the maximal open subscheme of $Hilb^n([U_i/G_i])$ where the rational map $Hilb^n([U_i/G_i]) \dashrightarrow Hilb^n(\mathscr{X})$ is defined.
Show that $W \to Hilb^n(\mathscr{X})$ is étale.
Then show that $Hilb^n([U_i/G_i])$ is smooth. $\square$

Proposition

Let $\mathscr{X}$ be a tame DM stack and is a global quotient. Suppose its coarse moduli space X is a quasi-projective scheme.
Then there exists a morphism

$$\mathrm{Hilb}^n(\mathscr{X}) \to \mathrm{Sym}^n(X)$$

taking a zero-dimensional substack to the underlying set of points in X over which the substack is supported.

Remark: Neeman showed that $\mathrm{Sym}^n\mathbb{P}^m \to Chow_{0,n}\mathbb{P}^m$ is not an isomorphism if $\mathrm{char}\ k = p > 0$ and $n, m \geq p + 1$.

Theorem

Let $\mathscr{X}$ be a smooth 2-dim tame DM stack with a connected quasi-projective coarse moduli space X. Assume $\mathscr{X}$ has only isolated stacky points and each isotropy group is
(1) abelian, or,
(2) a subgroup of $SL(2, \mathbb{C})$ (for $k = \mathbb{C}$).
Then the quasi-projective scheme $\mathrm{Hilb}^n \mathscr{X}$ is connected.

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Idea: Consider $\pi : \mathrm{Hilb}^n(\mathscr{X}) \to \mathrm{Sym}^n(X)$. Since $\mathrm{Sym}^n(X)$ is connected, it suffices to show each fiber of π is connected. Each fiber is isomorphic to a fiber of

$$\mathrm{Hilb}^n([\mathbb{A}^2/G]) \to \mathrm{Sym}^n(\mathbb{A}^2/G).$$

By Zariski's main theorem, it suffices to show that $\mathrm{Hilb}^n([\mathbb{A}^2/G])$ is connected, which is known under condition (1) or (2). □

Example

For $\mathscr{X} = [\mathbb{A}^2/G]$, $\mathrm{Hilb}^n(\mathscr{X})$=Hilbert schemes of regular G-orbits.

- *G=abelian group: $\mathrm{Hilb}^n(\mathscr{X})$ is a multigraded Hilbert scheme.*
- *$n=1$: $\mathrm{Hilb}^1(\mathscr{X}) = G$-Hilbert schemes.*
- *$G \subset SL_2(\mathbb{C})$: $\mathrm{Hilb}^n(\mathscr{X})$ is a quiver variety.*

Proposition

Suppose $(a, m) = 1$, $1 \leq a \leq m - 1$.
μ_m *act on* $\mathbb{C}^2$ *as* $\omega \cdot (x, y) = (\omega x, \omega^a y)$ *where* $\omega = e^{2\pi i/m}$.
Let $\tilde{X}$ *be the minimal resolution of* $\mathbb{C}^2/\mu_m$.
Then the natural birational map

$$\mathrm{Hilb}^n(\tilde{X}) \dashrightarrow \mathrm{Hilb}^n([\mathbb{C}^2/\mu_m])$$

is not a morphism for $n \geq 2$.

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Remark: Take $a = m - 1$. Then $\mathrm{Hilb}^n(\tilde{X})$ and $\mathrm{Hilb}^n([\mathbb{C}^2/\mu_m])$ give different crepant resolutions of $\mathrm{Sym}^n(\mathbb{C}^2/\mu_m)$ for $n \geq 2$.

Betti numbers and cellular decomposition of $\mathrm{Hilb}^n([\mathbb{A}^2/\mu_m])$

Theorem

Let $n, a, b, m \in \mathbb{N}$ such that $gcd(a, b, m) = 1$. Let μ_m acting on $\mathbb{A}^2$ as $\omega(x, y) = (\omega^a x, \omega^b y)$ where $\omega^m = 1$. Then $\mathrm{Hilb}^n([\mathbb{A}^2/\mu_m])$ is a smooth irreducible quasi-projective scheme with a cellular decomposition, and this decomposition is described in terms of certain combinatoric data.

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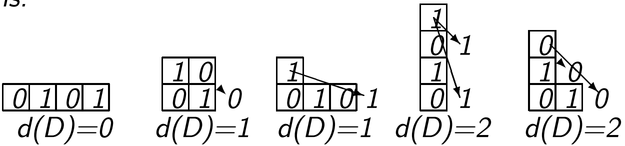
Example

For $n = 2, a = 1, b = 1, m = 2$, there are five admissible Young diagrams.



$\mathrm{Hilb}^2([\mathbb{A}^2/\mu_2]) \cong \mathbb{A}^4 + 2\mathbb{A}^3 + 2\mathbb{A}^2$, $\quad \mathrm{Hilb}^2_0([\mathbb{A}^2/\mu_2]) \cong \mathbb{A}^0 + 2\mathbb{A}^1 + 2\mathbb{A}^2$.

Summary:

- Define $\mathrm{Hilb}^n(\mathscr{X})$ for a tame DM-stack.
- Let $\mathscr{X}$ be a smooth tame DM-stack of dim 2,
  - $\mathrm{Hilb}^n(\mathscr{X})$ is smooth.
  - give sufficient conditions for $\mathrm{Hilb}^n(\mathscr{X})$ to be connected.
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Ongoing research:

- The global geometry of $\mathrm{Hilb}^n(\mathscr{X})$ for a toric DM stack $\mathscr{X}$.
- Connectedness of $\mathrm{Hilb}^n(\mathscr{X})$ for general $\mathscr{X}$.

Hilbert scheme and q, t-Catalan numbers (joint with Kyungyong Lee)

Consider the ideal $I = \cap_{1\leq i<j\leq n}(x_i - x_j, y_i - y_j)$ of $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.
Define $M = I/(\mathbf{x}, \mathbf{y})I$.

Problem (Haiman)

Find an explicit basis of the bi-graded vector space M.

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Theorem (Haiman)

1. $\dim M = \frac{1}{n+1}\binom{2n}{n}$.
2. *q, t-Catalan number* $C_n(q, t) = \sum_{d_1, d_2} t^{d_1} q^{d_2} \dim M_{d_1, d_2}$.
3. *Let H_0^n be the zero fiber of* $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$. *Then* $C_n(q, t) = \sum_{i=0}^{n-1} (-1)^i tr_{H^i(H_0^n, \mathcal{O}(1))}(q, t)$.

Table of q, t-catalan number for $n = 7$.

The coefficient of $q^{d_1} t^{d_2}$ is $p(k)$ for $k = n(n-1)/2 - d_1 - d_2, \quad d_1, d_2 \geq k$

1
0 1
0 1 1
0 1 1 1
0 1 2 1 1
0 1 2 2 1 1
0 1 3 3 2 1 1
0 0 2 4 3 2 1 1
0 0 2 4 5 3 2 1 1
0 0 1 4 5 5 3 2 1 1
0 0 1 3 6 6 5 3 2 1 1
0 0 0 2 5 7 6 5 3 2 1 1
0 0 0 1 4 6 8 6 5 3 2 1 1
0 0 0 0 2 5 7 8 6 5 3 2 1 1
0 0 0 0 1 3 6 8 8 6 5 3 2 1 1
0 0 0 0 0 1 3 6 7 8 6 5 3 2 1 1
0 0 0 0 0 0 1 3 5 6 7 6 5 3 2 1 1
0 0 0 0 0 0 0 1 2 4 5 6 5 5 3 2 1 1
0 0 0 0 0 0 0 0 0 1 2 3 4 4 4 3 2 1 1
0 0 0 0 0 0 0 0 0 0 0 1 1 2 2 3 2 2 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1

Theorem

Let d_1, d_2 be non-negative integers s.t. $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then

$$\dim M_{d_1,d_2} \leq p(\delta, k),$$

and the equality holds iff

- $k \leq n - 3$, *or*
- $k = n - 2$ *and* $\delta = 1$, *or*
- $\delta = 0$.

In case the equality holds, there is an explicit construction of a basis of M_{d_1,d_2}.

Idea to prove $\dim M_{d_1,d_2} \leq p(\delta, k)$

For any n-point set $D = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$
Define $\Delta(D) = \det[x_i^{\alpha_j} y_i^{\beta_j}]_{i,j}$, $bideg(D) := (\sum \alpha_j, \sum \beta_j)$.
Then $\{\Delta(D)\}_{bideg(D)=(d_1,d_2)}$ generates M_{d_1,d_2}.

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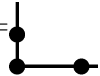
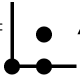
Example

3-point sets of bidegree $(2, 1)$:

$D =$  $\Delta(D) = \begin{vmatrix} 1 & x_1^2 & y_1 \\ 1 & x_2^2 & y_2 \\ 1 & x_3^2 & y_3 \end{vmatrix}$ $D' =$  $\Delta(D') = \begin{vmatrix} 1 & x_1 & x_1 y_1 \\ 1 & x_2 & x_2 y_2 \\ 1 & x_3 & x_3 y_3 \end{vmatrix}$

Then $\Delta(D)$, $\Delta(D')$ generate $M_{2,1}$.

But such generators are redundant in general.

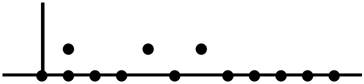
For a bidegree (d_1, d_2) satisfying

$$k := \binom{n}{2} - d_1 - d_2 << n,$$

there are unique integers a_μ such that

$$\Delta(D) = \sum a_\mu \Delta(F_\mu) \text{ in } M$$

where $F_\mu =$ is an n-point set of bidegree (d_1, d_2) and is of the form



(in the example, the partition type is $\mu = (2, 1, 5)$).
In other words, $\{\Delta(F_\mu)\}$ form a basis, for μ runs through partition of k into at most $\delta = \min(d_1, d_2)$ parts.
Therefore

$$\dim M_{d_1,d_2} \leq p(\delta, k).$$

Idea to prove $\dim M_{d_1,d_2} \geq p(\delta, k)$ **for** $k \leq n-3$**.**
For each D, by adding sufficient many points, we get $\tilde{D}$, such that $\Delta(\tilde{D})$ can be written uniquely as

$$\Delta(D) = \sum a_\mu \Delta(F_\mu) \text{ in } M.$$

Define

$$\varphi(\Delta(D)) := \sum a_\mu \rho_\mu,$$

where $\rho_\mu = \rho_{\mu_1}\rho_{\mu_2}\cdots\rho_{\mu_\ell}$ for $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$.

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where $\rho_\mu = \rho_{\mu_1}\rho_{\mu_2}\cdots\rho_{\mu_\ell}$ for $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$.

Define *weight*$(\rho_i) = i$. For $w \in \mathbb{Z}$, $f \in \mathbb{C}[[\rho_1, \rho_2, \ldots]]$, denote $\{f\}_w = \text{weight-}w \text{ part of } f$.

Lemma

$$\varphi(\Delta(D)) = (-1)^k \det\left[\left\{(1+\rho_1+\rho_2+\cdots)^{\beta_i}\right\}_{j-1-\alpha_i-\beta_i}\right]_{i,j}$$

Proposition

φ induces a linear map $\bar{\varphi} : M \to \mathbb{C}[\rho_1, \rho_2, \ldots]$.

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For each partition $\mu \in \Pi(\delta, k)$, we explicitly construct D_μ, s.t.

$$LM\ \bar{\varphi}(\Delta(D_\mu)) = \rho_\mu.$$

$\Rightarrow \{\bar{\varphi}(\Delta(D_\mu))\}$ are linearly independent

$\Rightarrow \{\Delta(D_\mu)\}$ are linearly independent (since $\bar{\varphi}$ is well-defined).

$\Rightarrow \dim M_{d_1,d_2} \geq p(\delta, k)$ for $k \leq n-3$. □

Summary:

- For $I = \cap(x_i - x_j, y_i - y_j)$, $M = I/(\mathbf{x}, \mathbf{y})I$ arises in the study of $Hilb^n(\mathbb{C}^2)$. Its Hilbert series gives the q, t-Catalan number.
- $\dim M_{d_1,d_2} \leq p(\delta, k)$, where $k = \binom{n}{2} - d_1 - d_2$, $\delta = \min(d_1, d_2)$.
- For $k \leq n - 3$, above "$=$" holds, we find an explicit basis for M_{d_1,d_2}.

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- For $I = \cap(x_i - x_j, y_i - y_j)$, $M = I/(\mathbf{x}, \mathbf{y})I$ arises in the study of $Hilb^n(\mathbb{C}^2)$. Its Hilbert series gives the q, t-Catalan number.
- $\dim M_{d_1,d_2} \leq p(\delta, k)$, where $k = \binom{n}{2} - d_1 - d_2$, $\delta = \min(d_1, d_2)$.
- For $k \leq n - 3$, above "$=$" holds, we find an explicit basis for M_{d_1,d_2}.

Ongoing research:

- Conjectural basis for M_{d_1,d_2}.
- Extend our method to the study of $I = \cap(x_i - x_j, y_i - y_j, z_i - z_j)$ and $\mathrm{Hilb}^n(\mathbb{C}^3)$.