8. Show that if $A$ is symmetric positive definite then $\det(A) > 0$. Give an example of a $2 \times 2$ matrix with positive determinant that is not positive definite.

Solution: (Joe)

We have that $A$ is positive definite. Let's say that $\lambda$ is an eigenvalue of $A$. Then, for any eigenvector $x$ belonging to $\lambda$ we have that

$$x^T A x = \lambda x^T x = \lambda ||x||^2$$

Thus,

$$\lambda = \frac{x^T A x}{||x||^2} > 0$$

So, all of the eigenvalues belonging to $A$ must be positive. The product of the eigenvalues of a matrix equals the determinant. Since all of the eigenvalues of $A$ are positive, their product must be positive, so $\det(A) > 0$, as desired.

The converse of what we just proved is not necessarily true. Take, for example, the negative identity matrix. Its determinant equals 1, but both of its eigenvalues are $-1$, so it is negative definite.

§6.4

11. Given

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

find a matrix $B$ such that $B^H B = A$.

Solution (Jeff) the characteristic equation for this matrix is:

$$\det \begin{pmatrix} 4 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & i \\ 0 & -i & 1 - \lambda \end{pmatrix} = 0.$$ 

This implies:

$$0 = (4 - \lambda)((1 - \lambda)^2 - 1)$$
$$= (4 - \lambda)((1 - \lambda) - 1)((1 - \lambda) + 1)$$
$$= (4 - \lambda)(-\lambda)(2 - \lambda).$$

Thus the eigenvalues are 4, 0, and 2. Then, from section 6.4 we know that there
is a matrix $U$ that diagonalizes $A$ which has the property that $A = UDU^H$. From the eigenvalues we know that matrix $D$ can be written as:

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

From chapter one we know that the first row of $U^H$ will be quadrupled. We also know the second row will be made all zeros and the third row will be doubled by the matrix $D$. We also know that $B^H B = UDU^H = A$. Thus is clearly follows that once we get $U^H$ we must multiply its first row by two, its second row by zero, and its third row by $2^{(1/2)}$ in order to do the job of $D$. Once this is done we will have the matrix $B$.

Upon doing simple calculation we see that a vector associated with the eigenvalue of one is $\frac{1}{2^{(1/2)}} \begin{pmatrix} 2^{1/2} \\ 0 \\ 0 \end{pmatrix}$. We also find that eigenvectors for the eigenvalues of zero and two are $\frac{1}{2^{(1/2)}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$ and $\frac{1}{2^{(1/2)}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ respectively. Thus the matrix $U$ is given by the following:

$$U = 1/\sqrt{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -i & i \\ 0 & 1 & 1 \end{pmatrix}.$$

Also:

$$U^H = 1/\sqrt{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & i & 1 \\ 0 & -i & 1 \end{pmatrix}.$$

This then clearly implies that:

$$B = 1/\sqrt{2} \begin{pmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & 0(i) & 0(1) \\ 0 & \sqrt{2}(i) & \sqrt{2}(1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 1 \end{pmatrix}.$$