§5.4
29. Consider the vector space $\mathbb{R}^n$ with inner product $\langle x, y \rangle = x^T y$. For any $n \times n$ matrix $A$, show that

a) $\langle Ax, y \rangle = \langle x, A^T y \rangle$

b) $\langle A^T Ax, x \rangle = \|Ax\|^2$

Solutions: (Jeff)

a) Using properties from the book we have:

$\langle Ax, y \rangle = (Ax)^T y$ (by the given)

= $x^T A^T y$ (transpose property)

= $x^T (A^T y)$ (associative property)

= $\langle x, A^T y \rangle$. (by the given)

b) Using the properties form the book we have:

$\langle A^T Ax, x \rangle = (A^T Ax)^T x$ (by the given)

= $x^T (A^T A)^T x$ (transpose property)

= $x^T (A^T (A^T)^T) x$ (transpose property)

= $x^T A^T A x$ (double transpose property)

= $((Ax)^T A)x$ (transpose property)

= $(Ax)^T (Ax)$ (associative property)

= $\langle Ax, Ax \rangle$. (by the given)

= $\|Ax\|^2$. (by the findings of section 5.1)

§5.5
30. Find the best least squares approximation to $f(x) = |x|$ on $[-\pi, \pi]$ by a trigonometric polynomial of degree less than or equal to 2.

Solution: (Joe)

According to Leon, the formula for a trigonometric approximation of degree 2 for a function is given by:

$$f_2(x) = \frac{a_0}{2} + \sum_{k=1}^{2} (a_k \cos kx + b_k \sin kx)$$

Also, we are shown that the following are the formulas for the coefficients of our degree 2 approximation:

$$a_0 = \langle |x|, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx$$
\[ a_1 = \langle |x|, \cos x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos x \, dx \]

\[ b_1 = \langle |x|, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin x \, dx \]

\[ a_2 = \langle |x|, \cos 2x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos 2x \, dx \]

\[ b_2 = \langle |x|, \sin 2x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin 2x \, dx \]

We know that \( b_1 \) and \( b_2 \) are odd functions, so over the interval \([-\pi, \pi]\) these will both equal 0. This leaves us to only need to compute \( a_0, a_1, \) and \( a_2 \). Since the integrals defined by the inner products in \( a_0, a_1, \) and \( a_2 \) are symmetric over the \( y \)-axis, we need only to take the following integrals over the interval \([0, \pi]\) and then multiply them by two:

\[ a_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \frac{2}{\pi} \frac{x^2}{2} \bigg|_0^\pi = \pi \]

\[ a_1 = \frac{2}{\pi} \int_0^\pi x \cos x \, dx = \frac{2}{\pi} \left[ x \sin x \right]_0^\pi - \int_0^\pi \sin x \, dx = \frac{2}{\pi} \cos x \bigg|_0^\pi = -\frac{4}{\pi} \]

\[ a_2 = \frac{2}{\pi} \int_0^\pi x \cos 2x \, dx = \frac{2}{\pi} \left[ \frac{x \sin x}{2} \right]_0^\pi - \int_0^\pi \frac{\sin 2x}{2} \, dx = \frac{2}{\pi} \cos 2x \bigg|_0^\pi = 0 \]

Now, using \( a_0, a_1, \) and \( a_2 \) in the aforementioned formula, we obtain

\[ f_2(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x \]

as our desired degree 2 trigonometric approximation of \( f(x) = |x| \).