19. Let $A$ and $B$ be $n \times n$ matrices.

a) Show that $AB = 0$ if and only if the column space of $B$ is a subspace of the nullspace of $A$

b) Show that if $AB = 0$, then the sum of the ranks of $A$ and $B$ cannot exceed $n$.

Solutions (Jeff):

a) For this part of the problem let the columns of $B$ be $b_1, b_2, \ldots, b_n$. Then $B = [b_1, b_2, \ldots, b_n]$. $AB = 0$ if and only if

$$AB = 0 \iff A[b_1, b_2, \ldots, b_n] = [Ab_1, Ab_2, \ldots, Ab_n] = 0 \iff \quad Ab_1 = Ab_2 = \cdots = Ab_n = 0$$

All the elements of the column space of $B$ must be contained in the nullspace of $A$.

b) We have already shown that if $AB = 0$ then the column space of $B$ is a subspace of the nullspace of $A$. Recall from the book that $\text{rank}(B) = \dim(\text{column space of } B)$. From the given we have that $\dim(\text{Null}(A)) \geq \dim(\text{rank}(B))$. This is true by the definition of dimension. We also know from theorem 3.6.5 that $\dim(\text{Null}(A)) + \dim(\text{rank}(A)) = n$ and $\dim(\text{Null}(B)) + \dim(\text{rank}(B)) = n$. Upon substitution we have $n - \dim(\text{rank}(A)) \geq \dim(\text{rank}(B))$. This yields $n \geq \dim(\text{rank}(A) + \dim(\text{rank}(B)))$ which is clearly what was desired.

22. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C = AB$. Show that

(a) The column space of $C$ is a subspace of the column space of $A$.
(b) The row space of $C$ is a subspace of the row space of $B$.
(c) $\text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

Solution: (Joe)

(a) By the definition of matrix multiplication, we have that the entries in $C$, where $C$ is the product $C = AB$, are defined by

$$c_{ij} = a(i,:)b_j = \sum_{k=1}^{n} a_{ik}b_{kj}$$

We can see that every column of $C$ is a linear combination of the columns of $A$, so the columns of $C$ lie in the subspace spanned by the columns of $A$. Therefore, the column space of $C$ is a subspace of the column space of $A$.

(b) We have that the rows of $C$ are the columns of $C^T$ and that $C^T = (AB)^T = B^TA^T$. Similarly to part (a), we then have that every column of $C^T$ is a linear combination of the columns of $B^T$, so the columns of $C^T$ lie in the subspace spanned by the columns of $B^T$. So, the rows of $C$ lie in the subspace
spanned by the rows of $B$. Therefore, the row space of $C$ is a subspace of the row space of $B$.

(c) Since the dimension of the column space equals to the rank, so (a) implies $\text{rank}(C) \leq \text{rank}(A)$. Similarly $\text{rank}(C) \leq \text{rank}(B)$. Therefore, we know that $\text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

§4.1

20. Let $L : V \rightarrow W$ be a linear transformation, and let $T$ be a subspace of $W$. The inverse image of $T$, denoted $L^{-1}(T)$ is defined by

$$L^{-1}(T) = \{v \in V : L(v) \in T\}$$

show that $L^{-1}(T)$ is a subspace of $V$.

Solution (Jeff):

Since $T$ is a subspace of $W$ it is clear that $L^{-1}(T)$ is a subset of $V$. This is because some of the elements in $V$ make up all the elements of $L^{-1}(T)$. Now, in order to show that $L^{-1}(T)$ is a subspace of $V$ we must also show that:

i) $\alpha x \in L^{-1}(T)$ for any $x$ in $L^{-1}(T)$ and any scalar $\alpha$.

ii) $x + y \in L^{-1}(T)$ for any $x$ and $y$ in $L^{-1}(T)$.

Now since $L$ is a linear transformation we know that $\alpha L(x) = L(\alpha x)$ for any $x \in V$. Also, since $T$ is a subspace of $W$ we know that $L(x) \in T$ implies that $\alpha L(x) \in T$ and therefore $L(\alpha x) \in T$. In other words $\alpha x \in L^{-1}(T)$ for any $x$ in $L^{-1}(T)$ and any scalar $\alpha$ (by the given definition).

Also, since $L$ is a linear transformation we know that $L(x + y) = L(x) + L(y)$ for any $x, y \in V$. Also, since $T$ is a subspace of $W$ we know that $L(x) \in T$ and $L(y) \in T$ implies that $L(x) + L(y) \in T$ and $L(x + y) \in T$. Thus if we have two elements $x, y \in V$ we know that $L(x + y) \in T$. In other words $x + y \in L^{-1}(T)$ for any $x$ and $y$ in $L^{-1}(T)$ (by the given definition). This completes the proof.

21. A linear transformation $L : V \rightarrow W$ is said to be one-to-one if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (i.e. no two distinct vectors $\mathbf{v}_1$, $\mathbf{v}_2$ in $V$ get mapped into the same vector $\mathbf{w} \in W$). Show that $L$ is one-to-one if and only if $\text{ker}(L) = \{0\}$. 

Solution: (Joe)

We must show that "If $L$ is one-to-one, then $\text{ker}(L) = \{0\}$" and that "if $\text{ker}(L) = \{0\}$, then $L$ is one-to-one."
First, suppose that $L$ is one-to-one. Then, there is at most one vector that maps to $0_W$. But $0_V$ maps to $0_W$ since $L$ is a linear transformation, so we have that $\text{ker}(L) = \{0_V\}$.

Second, suppose that $\text{ker}(L) = \{0_V\}$. Let us have two vectors $v_1, v_2$ that map to an element of $W$ such that $L(v_1) = L(v_2)$. Then, $L(v_1) - L(v_2) = 0_V$, so it follows that $L(v_1 - v_2) = 0_V$. Since $\text{ker}(L) = \{0_V\}$, we must have that $v_1 - v_2 = 0_V$. Hence, $v_1 = v_2$, and $L$ is one-to-one.

Combining the two parts, we have that $L$ is one-to-one if and only if $\text{ker}(L) = \{0_V\}$. 

\[3\]