Final Exam
MATH 415, B13, B14 - Linear Algebra
Fall 2007

Dec. 10, 2007

Name: __________________________________________

This exam has 8 questions, for a total of 123 points. There are 13 bonus points(problem 6(d) and problem 8).
Please show ALL your work.

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points:</td>
<td>15</td>
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<td>18</td>
<td>20</td>
<td>10</td>
<td>123</td>
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<td>Score:</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Score: 1
1. (a) (10 points) Find the inverse of the matrix

\[ A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 2 & 4 & 2 \end{pmatrix} \]

Solution:

\[ \begin{pmatrix} -2 & -5 & 3 \\ 1 & 3 & -3/2 \\ 0 & -1 & 1/2 \end{pmatrix} \]

(b) (5 points) Use the answer above to solve the system of equations

\[
\begin{align*}
    x_2 + 3x_3 &= 1 \\
    x_1 + 2x_2 &= 1 \\
    2x_1 + 4x_2 + 2x_3 &= 2
\end{align*}
\]

Solution:

\[ \begin{pmatrix} -2 & -5 & 3 \\ 1 & 3 & -3/2 \\ 0 & -1 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \]
2. (a) (12 points) Factor the matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ into a product $XDX^{-1}$, where $D$ is diagonal. (You do not need to calculate $X^{-1}$.)

(b) (3 points) What are the eigenvalues of $A^3$?

**Solution:** (a) there are three eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Corresponding eigenvectors are

$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}

(this answer is not necessarily unique) So

$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$

(b) They are $1, 2^3, 3^3$ (or $1, 8, 27$).
3. (15 points) Let $L$ be the linear operator on $P_3$ (the space of polynomials of degree $3$) defined by $L(p(x)) = xp''(x) - 2p(x)$. Consider two basis of $P_3$: the standard basis $[1, x, x^2]$, and the basis $[1, 1 + x, 1 + x + x^2]$.

(a) Find the transition matrix from $[1, 1 + x, 1 + x + x^2]$ to $[1, x, x^2]$.

(b) Find the matrix $A$ representing $L$ with respect to $[1, x, x^2]$.

(c) Find the matrix $B$ representing $L$ with respect to $[1, 1 + x, 1 + x + x^2]$.

**Solution:**

(a) $S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Since $L(1) = -2, L(x) = -2x, L(x^2) = 2x - 2x^2$, so

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

(c) $B = S^{-1}AS = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$. 

Page 4 Please go on to the next page...
4. (15 points) Let $P_4$ be the vector space of polynomials of degree $< 4$.

(a) Find the dimension of the subspace $V$ spanned by 4 polynomials:

$$V = \text{Span}(x^3 + 2x^2 - x, 2x^3 + 4x^2 - 2x, 2x^3 + 5x^2 - 3x + 2, 3x^3 + 8x^2 - 5x + 4)$$

(b) Moreover, find a basis for $V$.

**Solution:** Take the standard basis $[x^3, x^2, x, 1]$ for $P_4$, the subspace $V$ is the same as the column space of the matrix

$$
\begin{pmatrix}
1 & 2 & 2 & 3 \\
2 & 4 & 5 & 8 \\
-1 & -2 & -3 & -5 \\
0 & 0 & 2 & 4
\end{pmatrix}
$$

The row echelon form is

$$
\begin{pmatrix}
1 & 2 & 2 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

Since there are two leading 1’s, so

(a) The dimension of $V$ is 2.

(b) A basis is the first two polynomials: $x^3 + 2x^2 - x, 2x^3 + 5x^2 - 3x + 2$. 

Page 5 Please go on to the next page...
5. (a) (12 points) For the quadratic equation \(2xy + 2xz - 2yz = 1\), find a change of coordinates so that the resulting conic section is in standard form.

**Solution:** The corresponding symmetric matrix is

\[
A = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & -1 \\
1 & -1 & 0 \\
\end{pmatrix}
\]

\[
det(A - \lambda I) = -(\lambda - 1)^2(\lambda + 2)
\]

therefore \(A\) has eigenvalues 1, 1, -2. The corresponding eigenvectors are

\[
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
1 \\
-2
\end{pmatrix}, \begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix}
\]

So we can use the orthogonal matrix

\[
Q = \begin{pmatrix}
1\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\
1\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\
0 & -2/\sqrt{6} & 1/\sqrt{3}
\end{pmatrix}
\]

to diagonalize \(A\), the corresponding diagonal matrix

\[
D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]

So by changing the coordinates \(\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = Q^T \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}\), we get the standard form

\[
(x')^2 + (y')^2 - 2(z')^2 = 1
\]

(b) (3 points) For the function \(f(x, y, z) = 2xy + 2xz - 2yz\), determine whether the stationary point (0, 0) corresponds to a local maximal, local minimum, or saddle point.

**Solution:** Since there are positive and negative eigenvalues, the point (0, 0) is a saddle point.
6. (15 points) Let $\mathbb{R}^{2 \times 2}$ be the space of all $2 \times 2$ matrices. Let $L : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ be the linear transformation $L(M) = M - M^T$. (here $M^T$ means the transpose of $M$.)

(a) Find the matrix $A$ representing $L$ with respect to the standard basis $[e_1, e_2, e_3, e_4]$, where

\[
e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\]

(b) Find a basis for the kernel of $L$, and determine its dimension.

(c) Find a basis for the range of $L$, and determine its dimension.

(d) (3 points) * In general, consider the linear transformation $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ defined by $f(M) = M - M^T$. Determine the dimension, in terms of $n$, of the kernel of $f$.

**Solution:**

(a) Since $L(e_1) = 0, L(e_2) = e_2 - e_3, L(e_3) = e_3 - e_2, L(e_4) = 0$, so the matrix representing $L$ is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(b) A basis for the kernel of $L$ is $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 + e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Its dimension is 3.

(c) A basis for the range of $L$ is $e_2 - e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Its dimension is 1.

(d) The kernel is the set of symmetric matrices, it is all determined by the upper triangular part, so the dimension is

\[n + (n - 1) + \cdots + 1 = \frac{n(n + 1)}{2} \, .\]
7. (20 points) Let $V$ be a subspace of $\mathbb{R}^3$ spanned by two vectors $x_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

(a) Use the Gram-Schmidt process to find an orthonormal basis for $V$, and find a QR-factorization of $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ -2 & 2 \end{pmatrix}$.

Solution: $r_{11} = \|x_1\| = 3$, and $q_1 = x_1/r_{11} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$;

$r_{12} = \langle q_1, x_2 \rangle = -1$, $r_{22} = \|x_2 - r_{12}q_1\| = \| \begin{pmatrix} 2/3 \\ 4/3 \\ 4/3 \end{pmatrix} \| = 2$, $q_2 = x_2 - r_{12}q_1 = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$.

Then $q_1$ and $q_2$ will be an orthonormal basis for $V$. And the $QR$-factorization is

$$A = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \\ -2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}.$$

(b) Find the projection matrix $P$ that projects vectors in $\mathbb{R}^3$ onto $V$. (Hint: $P$ should be a $3 \times 3$ matrix.)

Solution: $P = QQ^T = 1/9 \begin{pmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{pmatrix}$.

(c) Find the vector $p$ in $V$ that is closest to $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: $Pw = 1/9 \begin{pmatrix} 7 \\ 11 \\ 8 \end{pmatrix}$. 

Page 8 Please go on to the next page...
8. (10 points) * Choose ONE of the following two problems to do:

(a) Let $B$ be a (real) symmetric positive definite matrix and also an orthogonal matrix. Show that $B$ is the identity matrix.

**Solution:**

1. Since $B$ is symmetric, so it can be diagonalized by orthogonal matrix: 
   
   $$B = QDQ^T.$$ 

2. $B$ is positive definite, so $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ has all eigenvalues $\lambda_1, \ldots, \lambda_n$ positive.

3. $B$ is also an orthogonal matrix, so $B^T B = I$, that is $(QDQ^T)(QDQ^T) = I$, which simplified to $QD^2Q^T = I$. From here we see that $D^2$ and $I$ have the same eigenvalues. But the eigenvalues for $D^2$ are $\lambda_1^2, \ldots, \lambda_n^2$. Then $\lambda_1 = \cdots = \lambda_n = 1$. This means $D = I$, and therefore $B = QDQ^T = QQ^T = I$.

(b) Find all the eigenvalues for the following $n \times n$ matrix 

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$ 

Then prove your conclusion. (Hint: try small $n$.)

**Solution:** It is enough to find the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 1 & \cdots & 1 \\ 1 & 1 - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \lambda \end{pmatrix}$$

(add all other rows to the first row)

$$= \det \begin{pmatrix} n - \lambda & n - \lambda & \cdots & n - \lambda \\ 1 & 1 - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \lambda \end{pmatrix} = (n - \lambda) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \lambda \end{pmatrix}$$

subtract the first row from all other rows:

$$= (n - \lambda) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda \end{pmatrix} = (n - \lambda)(-\lambda)^{n-1}.$$ 

So the eigenvalues are $n$ and $0$ (with multiplicity $n - 1$).