Saturation Number of Ramsey-Minimal Graphs for Matchings

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Definition

Given a family $\mathcal{F}$ of graphs, $G$ is $\mathcal{F}$-saturated if:

1. $G$ contains no member of $\mathcal{F}$, and
2. for any pair of nonadjacent vertices $u$ and $v$ in $G$, $G + uv$ contains some member of $\mathcal{F}$.

If $\mathcal{F} = \{\mathcal{F}\}$, we then say that $G$ is $\mathcal{F}$-saturated.
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2. for any pair of nonadjacent vertices $u$ and $v$ in $G$, $G + uv$ contains some member of $\mathcal{F}$.

If $\mathcal{F} = \{F\}$, we then say that $G$ is $F$-saturated.
The Turán Problem

**Problem (The Turán Problem)**

Determine \( \text{ex}(n, \mathcal{F}) \), the maximum number of edges in a graph that contains no member of \( \mathcal{F} \) as a subgraph.

\( \text{ex}(n, \mathcal{F}) \) is the **extremal** or **Turán** number of \( \mathcal{F} \).
\( \text{ex}(n, \mathcal{F}) \) is the maximum number of edges in an \( \mathcal{F} \)-saturated graph.
$\text{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-saturated graph.

**Definition**

The *minimum* number of edges in an $\mathcal{F}$-saturated graph is denoted $\text{sat}(n, \mathcal{F})$. 
Erdős, Hajnal and Moon determined $sat(n, K_r)$ exactly.

**Theorem (E-H-M 1964)**

\[ sat(n, K_r) = e(K_{r-2} + \overline{K}_{n-r+2}) = \binom{r-2}{2} + (r - 2)(n - r + 2). \]

*Furthermore, $K_r + K_{n-r+2}$ is the unique $K_r$-saturated graph of minimum size.*
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**Theorem (E-H-M 1964)**

$$\text{sat}(n, K_r) = e(K_{r-2} + \overline{K}_{n-r+2}) = \binom{r-2}{2} + (r - 2)(n - r + 2).$$

*Furthermore, $K_r + K_{n-r+2}$ is the unique $K_r$-saturated graph of minimum size.*
Interestingly, \( \text{sat}(n, \mathcal{F}) \) does not share many of the nice properties of \( \text{ex}(n, \mathcal{F}) \).

\[
\text{sat}(n, F) \not\leq \text{sat}(n + 1, F)
\]

\[
\text{sat}(2k - 1, P_4) = k + 1 > \text{sat}(2k, P_4) = k
\]

\[
F' \subset F \not\Rightarrow \text{sat}(n, F') \leq \text{sat}(n, F)
\]

\[
\text{sat}(n, K_{1,m}) > \text{sat}(n, K_{1,m} + e)
\]

\[
\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow \text{sat}(n, \mathcal{F}_1) \geq \text{sat}(n, \mathcal{F}_2)
\]

\[
\text{sat}(n, \{K_{1,m} + e\}) < \text{sat}(n, \{K_{1,m}\}) = \text{sat}(n, \{K_{1,m}, K_{1,m} + e\})
\]
Some Known Results

\( \text{sat}(n, H) \) has been studied for many classes of graphs.

- \( K_{1,t} \) and \( P_t \) (Kásonyi and Tuza 1986)
- Matchings (Mader 1973, Kásonyi and Tuza 1986)
- \( tK_r \) and \( K_r \cup K_s \) (Faudree, Ferrara, Gould and Jacobson 2009)
- Trees (Faudree, Faudree, Gould, Jacobson 2009)
- A Survey of Minimum Saturated Graphs (Faudree, Faudree, Schmitt - submitted)
Bounding the \textit{sat} Function

**Theorem (Erdős-Stone-Simonovits)**

If $H$ is a nontrivial graph, then

\[
\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \frac{n^2}{2} + o(n^2).
\]

Specifically, if $H$ is not bipartite,

\[
\text{ex}(H, n) = \Theta(n^2).
\]
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Specifically, if $H$ is not bipartite,

$$ex(H, n) = \Theta(n^2).$$

**Theorem (Kásonyi and Tuza 1986)**

$$sat(n, \mathcal{F}) = O(n).$$
Every large enough $H$-saturated graph $G$ has

$$\delta(G) \geq \delta(H) - 1,$$

so

$$\text{sat}(n, H) \geq \frac{\delta(H) - 1}{2} n.$$
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$$sat(n, H) \geq \frac{\delta(H) - 1}{2} n.$$

**Problem**

*For an arbitrary graph $F$ determine a non-trivial lower bound on $sat(n, F)$.***
Ramsey Numbers

Definition

The **Ramsey Number** $r(H_1, H_2, \ldots, H_k)$ is the smallest integer $n$ such that every $k$-edge coloring of $K_n$ contains a copy of $H_i$ in color $i$ for some $i$. For instance, $r(K_3, K_3) = 6$. 

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![Graph showing a Ramsey configuration](image-url)
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![Ramsey Number Graph](image-url)
Why Only Complete Graphs?

**Definition**

Given graphs $G, H_1, \ldots, H_k$, we write

$$G \hookrightarrow (H_1, H_2, \ldots, H_k)$$

if every $k$-edge coloring of $G$ contains a copy of $H_i$ in color $i$ for some $i$. 

Therefore, $r(H_1, \ldots, H_k)$ is the smallest $n$ such that $K_n - 1 \not\hookrightarrow (H_1, \ldots, H_k)$, and $K_n \hookrightarrow (H_1, \ldots, H_k)$. 

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Therefore, $r(H_1, \ldots, H_k)$ is the smallest $n$ such that

$$K_{n-1} \not\hookrightarrow (H_1, \ldots, H_k),$$

and

$$K_n \hookrightarrow (H_1, \ldots, H_k).$$
In 1987, Hanson and Toft posed the following problem:

**Problem**

Determine the minimum number of edges in a graph \( G \) of order \( n \) such that

\[
G \not\rightarrow (K_{t_1}, \ldots, K_{t_k})
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but for any \( uv \in \overline{G} \),

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G + uv \rightarrow (K_{t_1}, \ldots, K_{t_k}).
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**Problem**

Let $t_1, \ldots, t_k$ be positive integers. Determine the minimum number of edges in a graph $G$ of order $n$ such that:

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1. there is a $k$-edge coloring of $G$ with no monochromatic $K_{t_i}$ in color $i$ for any $i$, and
2. for any $uv \in \overline{G}$, every $k$-edge-coloring of $G + uv$ contains a monochromatic copy of $K_{t_i}$ in color $i$ for some $i$. 
Definition

Given graphs $G, H_1, \ldots, H_k$, we say that $G$ is $(H_1, \ldots, H_k)$-Ramsey-minimal if $G \rightarrow (H_1, \ldots, H_k)$, but $G' \not\rightarrow (H_1, \ldots, H_k)$ for any proper subgraph $G'$ of $G$. 

Example: $C_5$ is $(P_3, P_3)$-Ramsey-minimal.
**Definition**

Given graphs $G, H_1, \ldots, H_k$, we say that $G$ is $(H_1, \ldots, H_k)$-Ramsey-minimal if $G \not\rightarrow (H_1, \ldots, H_k)$, but $G' \not\rightarrow (H_1, \ldots, H_k)$ for any proper subgraph $G'$ of $G$.

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(\(H_1, \ldots, H_k\))-Ramsey Minimality

**Definition**

Given graphs \(G, H_1, \ldots, H_K\), we say that \(G\) is \((H_1, \ldots, H_k)\)-Ramsey-minimal if \(G \rightrightarrows (H_1, \ldots, H_k)\), but \(G' \nleftrightarrow (H_1, \ldots, H_k)\) for any proper subgraph \(G'\) of \(G\).

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(\(H_1, \ldots, H_k\))-Ramsey Minimality

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Given graphs \(G, H_1, \ldots, H_k\), we say that \(G\) is **(\(H_1, \ldots, H_k\))-Ramsey-minimal** if \(G \hookrightarrow (H_1, \ldots, H_k)\), but \(G' \not\hookrightarrow (H_1, \ldots, H_k)\) for any proper subgraph \(G'\) of \(G\).

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\[ G \hookrightarrow (H_1, \ldots, H_k) \text{ if and only if } G \text{ contains an } (H_1, \ldots, H_k)\text{-Ramsey-minimal subgraph.} \]
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**Problem (Hanson and Toft)**

Determine the minimum number of edges in a graph \( G \) such that

\[ G \not\leftrightarrow (K_{t_1}, \ldots, K_{t_k}) \]

but for any \( uv \in \overline{G} \),

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An Important Observation:

Let $\mathcal{R}_{\text{min}}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_K)$-Ramsey minimal graphs.

Problem (Hanson and Toft)

Determine $\text{sat}(n, \mathcal{R}_{\text{min}}(K_{t_1}, \ldots, K_{t_k}))$. 
Conjecture (Hanson and Toft 1987)

Let \( r = r(K_{t_1}, \ldots, K_{t_k}) \). Then

\[
sat(n, R_{\min}(K_{t_1}, \ldots, K_{t_k})) = sat(n, K_r).
\]
The Hanson-Toft Conjecture

Conjecture (Hanson and Toft 1987)

Let \( r = r(K_{t_1}, \ldots, K_{t_k}) \). Then

\[
\text{sat}(n, R_{\text{min}}(K_{t_1}, \ldots, K_{t_k})) = \text{sat}(n, K_r).
\]

If \( t_i \geq 3 \) for at most one \( i \), the conjecture follows from Erdős-Hajnal-Moon.
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Let $r = r(K_{t_1}, \ldots, K_{t_k})$. Then

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Theorem (Chen, Ferrara, Gould, Magnant, Schmitt 2011)

For $n \geq 56$,

$$\text{sat}(n, R_{\text{min}}(K_3, K_3)) = \text{sat}(n, K_6) = 4n - 10.$$
Most simple case?
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**Fact**

\[ sat(n, R_{\text{min}}(K_2, K_2, \ldots, K_2)) = 0. \]
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**Problem**

\[ \text{sat}(n, R_{\text{min}}(m_1 K_2, m_2 K_2, \ldots, m_k K_2))? \]
Main Result

Theorem

For $n > 3(m_1 + \cdots + m_k - k)$,

$$
\text{sat}(n, R_{\text{min}}(m_1K_2, m_2K_2, \cdots, m_kK_2)) = 3(m_1 + m_2 + \cdots + m_k - k)
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If $m_i \leq 2$ for all $i$, the extremal graph is union of edge disjoint triangles and isolated vertices. Otherwise, the unique extremal graph is union of vertex disjoint triangles and isolated vertices.

Upper bound
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**Upper bound**

No monochromatic $2K_2$.

If we add an edge,
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Upper bound

\begin{figure}
\centering
\begin{tikzpicture}
\begin{scope}
\draw (0,0) -- (1,0) -- (0.5,0.866) -- (0,0);
\draw[red] (1,0) -- (2,0) -- (1.5,0.866) -- (1,0);
\draw[blue] (2,0) -- (3,0) -- (2.5,0.866) -- (2,0);
\draw[green] (3,0) -- (4,0) -- (3.5,0.866) -- (3,0);
\end{scope}
\end{tikzpicture}
\end{figure}
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For \( n > 3(m_1 + \cdots + m_k - k) \),

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We get a monochromatic \( 2K_2 \), in any case.
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**Upper bound**

We get a monochromatic \( 2K_2 \), in any case.
Proposition

$G$ has $n$ vertices, and $(m_1K_2, \cdots, m_kK_2)$-saturated. Then it contains at least $3(m_1 + m_2 + \cdots + m_k - k)$ edges.

Suppose we have a counterexample $G$. $G$ has less than $3(m_1 + m_2 + \cdots + m_k - k)$ edges and $(m_1K_2, \cdots, m_kK_2)$-saturated. It has a coloring avoiding monochromatic matchings.
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![Diagram of graphs showing the change in color of edges]
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![Diagram showing edge coloring changes](image_url)
First, we change color of an edge into red as long as it does not create any $m_r K_2$.

Once we cannot do it anymore, we get a red subgraph which is $m_1 K_2$ saturated. We call this subgraph as a ‘red-heavy’ subgraph.
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Once we cannot do it anymore, we get a red subgraph which is $m_1 K_2$ saturated. We call this subgraph as a ‘red-heavy’ subgraph.

We have this colored subgraph, then we start to change color of every edge into blue if possible to get a blue-heavy subgraph.
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Once we cannot do it anymore, we get a red subgraph which is $m_1 K_2$ saturated. We call this subgraph as a ‘red-heavy’ subgraph.

We have this colored subgraph, then we start to change color of every edge into blue if possible to get a blue-heavy subgraph. We do this for all color in order.
Theorem (Mader, 1973)

If $G$ is $mK_2$-saturated, and $n \geq 2m - 1$, then one of the following holds.
1. Every component of $G$ is an odd clique.
2. $G$ has a dominating vertex $v$ and $G - v$ is $(m - 1)K_2$-saturated.
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1. Every component of $G$ is an odd clique.
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Dominating vertices give too many edges, so $G$ does not have any dominating vertex.
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1. Every component of $G$ is an odd clique.
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Dominating vertices give too many edges, so $G$ does not have any dominating vertex. Thus each color-heavy subgraph is disjoint union of odd cliques. However, $G$ may be not disjoint union of odd cliques, since some edge might change colors.
Lemma

Any edge $e$ in $G$ belongs to at most two color-heavy subgraph.
Color heavy graphs

**Lemma**

*Any edge e in G belongs to at most two color-heavy subgraph.*
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We add an edge, and recolor it.
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Lemma

Any edge $e$ in $G$ belongs to at most two color-heavy subgraphs.

It does not contain a bigger matching, so $G$ is not saturated, a contradiction.
Lower bound

In addition, any edge in a $K_3$ component or $K_5$ component in a color-heavy subgraph belongs to only one color-heavy subgraph.
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For $k$ colors, we take 1st color-heavy subgraph, and we change color of edges to 2nd color until we cannot anymore so that we get 2nd color-heavy subgraph.
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For $k$ colors, we take 1st color-heavy subgraph, and we change color of edges to 2nd color until we cannot anymore so that we get 2nd color-heavy subgraph. We keep do this until we get $k$th color-heavy subgraph.
In addition, any edge in a $K_3$ component or $K_5$ component in a color-heavy subgraph belongs to only one color-heavy subgraph.

For $k$ colors, we take 1st color-heavy subgraph, and we change color of edges to 2nd color until we cannot anymore so that we get 2nd color-heavy subgraph. We keep do this until we get $k$th color-heavy subgraph.

Each color-heavy subgraph is $K_{2t_{i,1}+1} \cup K_{2t_{i,2}+1} \cup \cdots \cup K_{2t_{i,s(i)}+1}$ with $t_{i,1} + t_{i,2} + \cdots + t_{i,s(i)} = m_i - 1$. 


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For each color-heavy subgraph, we count 1 for each edge in $K_3$ or $K_5$ and we count $\frac{1}{2}$ for each edge in other components, then we count each edge in $G$ exactly once.
If \( t_{i,j} = 1 \), we get \( \binom{3}{2} = 3t_{i,j} \) edges.
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If $t_{i,j} \geq 3$, we get $\frac{1}{2} \left( \frac{2t_{i,j}+1}{2} \right) > 3t_{i,j}$ edges.
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If $t_{i,j} \geq 3$, we get $\frac{1}{2} \cdot \binom{2t_{i,j}+1}{2} > 3t_{i,j}$ edges.

In total, we get at least $3\left(\sum t_{i,j}\right) = 3(m_1 + m_2 + \cdots + m_k - k)$ edges, and we get a contradiction.
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In addition, having equality means we have bunch of $K_3$ only.
Lower bound

If $t_{i,j} = 1$, we get $\binom{3}{2} = 3t_{i,j}$ edges.
If $t_{i,j} = 2$, we get $\binom{5}{2} = 10 = 5t_{i,j}$ edges.
If $t_{i,j} \geq 3$, we get $\frac{1}{2}(2^{t_{i,j}+1}) > 3t_{i,j}$ edges.

In total, we get at least $3(\sum t_{i,j}) = 3(m_1 + m_2 + \cdots + m_k - k)$ edges, and we get a contradiction.
In addition, having equality means we have bunch of $K_3$ only.

If $m_i \geq 3$ for at least one $i$, they are vertex disjoint. If $m_i \leq 2$ for all $i$, they are just edge-disjoint.