Math-484 Homework #4 (repetition and A-G inequality)

I will finish the homework before 10am Sep 26. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am D14 (4 hours student).

1: (A little test repetition)
Define \( f(x, y, z) \) on \( \mathbb{R}^3 \) as \( f(x, y, z) = e^x + e^y + e^z + 2e^{-x-y-z} \). Show that \( Hf(x, y, z) \) is positive definite at all points of \( \mathbb{R}^3 \). Find strict global minimizer of \( f \).
*Hint:* You should get \( \left( \frac{\ln 2}{4}, \frac{\ln 2}{4}, \frac{\ln 2}{4} \right) \) as the minimizer.

2: (A little test repetition)
Show that no matter what value of \( a \) is chosen, the function \( f(x_1, x_2) = x_1^3 - 3ax_1x_2 + x_2^3 \) has no global maximizers. Determine the nature of the critical points of this function for all values of \( a \).

3: (I will recall convexity of a function)
Show that for all positive \( x \) and \( y \):
\[
\frac{x}{4} + \frac{3y}{4} \leq \sqrt{\ln \left( \frac{e^{x^2}}{4} + \frac{3e^{y^2}}{4} \right)}
\]
*Hint:* The desired inequality follows from convexity of an appropriate function.

4: (Can I use (A − G) inequality?)
Solve using \((A − G)\) inequality the following problems:
a) Minimize \( x^2 + y + z \) subject to \( xyz = 1 \) and \( x, y, z > 0 \)
b) Maximize \( xyz \) subject to \( 3x + 4y + 12z = 1 \) and \( x, y, z > 0 \)
c) Minimize \( 3x + 4y + 12z \) subject to \( xyz = 1 \) and \( x, y, z > 0 \)

5: (I wanna be a (GP) master!)
Solve the following \((GP)\) where \( c_1, c_2, c_3 \) are positive numbers:
Minimize \( f(x, y) = c_1x + c_2x^{-2}y^{-3} + c_3y^4 \) over all \( x, y > 0 \).

6: (Semidefinite matrices theoretically. D14 only)
Show that the matrix
\[
A(x) = \begin{pmatrix}
x^4 & x^3 & x^2 \\
x^3 & x^2 & x \\
x^2 & x & 1
\end{pmatrix}
\]
is positive semidefinite for all \( x \in \mathbb{R} \).
*Hint:* See page 79, ex. 13 and 14.