Math-484 Homework #7
I will finish this homework before 11 am Oct 19 and bring it to class. If I have troubles with my work I may come to the study session on Oct 17, 5-7 pm, 145 Altgeld Hall. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am D14 (4 hours student).

Exercise 1: (Can I work with H-norm?)
Minimize $f(x, y, z) = 2x^2 + 2xy + 2y^2 + z^2$
subject to $x - y + z = 3$
$9x + 6y + 2z = 5$

Exercise 2: (I want to know what if epigraph.)
Let $D \subset \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$. The epigraph of $f$ is a subset of $\mathbb{R}^{n+1}$ defined by
\[
\text{epi}(f) = \{(x, \alpha) : x \in D, \alpha \in \mathbb{R}, f(x) \leq \alpha\}.
\]
a) Sketch the epigraphs of functions
$f(x) = e^x$ for $x \in \mathbb{R}$
$f(x_1, x_2) = x_1^2 + x_2^2$ for $(x_1, x_2) \in \mathbb{R}^2$
b) Show that $f(x)$ is convex if and only if $\text{epi}(f)$ is convex.
c) Show that if $f(x)$ and $g(x)$ are convex functions defined on a convex set $C$ then $h(x) := \max\{f(x), g(x)\}$ is also a convex function on $C$ by showing that
\[
\text{epi}(\max\{f(x), g(x)\}) = \text{epi}(f(x)) \cap \text{epi}(g(x)).
\]

Exercise 3: (More about orthogonal complements)
Let $M$ be a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement $M^\perp$ of $M$ is closed.

Exercise 4: (What is an interior?)
Prove that if $M$ is a subspace of $\mathbb{R}^n$ such that $M \neq \mathbb{R}^n$, then the interior $M^0$ of $M$ is empty.

Exercise 5: (What are closest vectors?)
Let $C$ be a closed convex subset of $\mathbb{R}^n$. If $y \notin C$, show that $x^* \in C$ is the closest vector to $y$ in $C$ if and only if $(x - y) \cdot (x^* - y) \geq ||x^* - y||^2$ for all $x \in C$.

Exercise 6: (Do I remember ancient stuff?)
Show that
\[
\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4
\]
holds with equality if and only if $x = y = z = w$.

Exercise 7: (Can I separate things? D14 only)
Let $C_1, C_2 \subset \mathbb{R}^n$ be both convex, $C_1^0 \neq \emptyset$ and $C_1^0 \cap C_2 = \emptyset$. Prove that there exist $0 \neq a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that
\[
x^T a \leq \alpha \leq y^T a
\]
for all $x \in C_1$ and $y \in C_2$. In other words, there is a hyperplane separating $C_1$ and $C_2$ (but both $C_1$ and $C_2$ can intersect the hyperplane).