I. Find all the critical points of the given functions, and classify each as a relative maximum, a relative minimum, or a saddle point.

1. \( f(x, y) = x^2 + y^2 \)

2. \( f(x, y) = y^2 - x^2 \)

3. \( f(x, y) = 6 - y^2 - x^2 \)

4. \( f(x, y) = x^2 + 2y^2 - xy + 14y \)

5. \( f(x, y) = (x - 1)^2 + y^3 - 3y^2 - 9y - 9 \)

6. \( f(x, y) = \frac{16}{x} + \frac{6}{y} + x^2 - 3y^2 \)

7. \( f(x, y) = 2x^3 + y^3 + 3x^2 - 3y - 12x - 4 \)

8. \( f(x, y) = x^2 - 6xy - 2y^3 \)

9. \( f(x, y) = e^{-(x^2+y^2-6y)} \)

10. \( f(x, y) = (x - 4) \ln(xy) \)
I. Find all the critical points of the given functions, and classify each as a relative maximum, a relative minimum, or a saddle point.

1. \( f(x, y) = x^2 + y^2 \)
   Answer:
   \[
   f_x(x, y) = 2x \\
   f_y(x, y) = 2y \\
   f_x = f_y = 0 \text{ when } (x, y) = (0, 0) \\
   D = f_{xx}f_{yy} - (f_{xy})^2 \\
   D = (2)(2) - 0^2 \\
   D(0, 0) > 0 \text{ and } f_{xx} > 0 \text{ Therefore there is a relative minimum at } (x, y) = (0, 0).
   
2. \( f(x, y) = y^2 - x^2 \)
   Answer:
   \[
   f_x(x, y) = -2x \\
   f_y(x, y) = 2y \\
   f_x = f_y = 0 \text{ when } (x, y) = (0, 0) \\
   D = f_{xx}f_{yy} - (f_{xy})^2 \\
   D = (-2)(2) - 0^2 \\
   D(0, 0) < 0. \text{ Therefore there is a saddle point at } (x, y) = (0, 0).
   
3. \( f(x, y) = 6 - y^2 - x^2 \)
   Answer:
   \[
   f_x(x, y) = -2x \\
   f_y(x, y) = -2y \\
   f_x = f_y = 0 \text{ when } (x, y) = (0, 0) \\
   D = f_{xx}f_{yy} - (f_{xy})^2 \\
   D = (-2)(-2) - 0^2 \\
   D(0, 0) > 0 \text{ and } f_{xx} < 0 \text{ Therefore there is a relative maximum at } (x, y) = (0, 0).
   
4. \( f(x, y) = x^2 + 2y^2 - xy + 14y \)
   Answer:
\[ f_x(x, y) = 2x - y \]
\[ f_y(x, y) = 4y - x + 14 \]
\[ f_x = f_y = 0 \implies 2x = y \implies 8x - x + 14 \implies x = -2 \implies y = -4 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = (2)(4) - (-1)^2 \]
\[ D(0, 0) > 0 \text{ and } f_{xx} > 0 \text{ Therefore there is a relative minimum at } (x, y) = (-2, -4). \]

5. \[ f(x, y) = (x - 1)^2 + y^3 - 3y^2 - 9y - 9 \]
Answer:
\[ f_x(x, y) = 2(x - 1) \]
\[ f_y(x, y) = 3y^2 - 6y - 9 \]
\[ f_x = 0 \text{ when } x = 1 \]
\[ f_y = 3(y^2 - 2y - 3) = 3(y + 1)(y - 3) = 0 \text{ when } y = -1, 3 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = (2)(6y - 6) - 0^2 \]
\[ D(1, -1) < 0. \text{ Therefore there is a saddle point at } (x, y) = (1, -1). \]
\[ D(1, 3) > 0 \text{ and } f_{xx} > 0. \text{ Therefore there is a relative minimum at } (x, y) = (1, 3). \]

6. \[ f(x, y) = \frac{16}{x} + \frac{6}{y} + x^2 - 3y^2 \]
Answer:
\[ f_x(x, y) = -\frac{16}{x^2} + 2x \]
\[ f_y(x, y) = -\frac{6}{y^2} - 6y \]
\[ f_x = 0 \implies x = 2 \]
\[ f_y = 0 \implies y = -1 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = \left(\frac{32}{x^3} + 2\right)\left(\frac{36}{y^3} - 6\right) - (0)^2 \]
\[ D(2, -1) > 0 \text{ and } f_{xx} > 0 \text{ Therefore there is a relative minimum at } (x, y) = (2, 1). \]

7. \[ f(x, y) = 2x^3 + y^3 + 3x^2 - 3y - 12x - 4 \]
Answer:
\[ f_x(x, y) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1) \]
\[ f_y(x, y) = 3y^2 - 3 = 3(y - 1)(y + 1) \]
\[ f_x = 0 \text{ when } x = -2, 1 \]
\[ f_y = 0 \text{ when } y = -1, 1 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = (12x + 6)(6y) - 0^2 \]
\[ D(-2, -1) > 0 \text{ and } f_{xx}(-2, -1) < 0. \text{ Therefore there is a relative maximum at (}x, y\text{) = }(-2, -1).\]
\[ D(-2, 1) < 0. \text{ Therefore there is a saddle point at (}x, y\text{) = }(-2, 1).\]
\[ D(1, -1) < 0. \text{ Therefore there is a saddle point at (}x, y\text{) = }(-1, -1).\]
\[ D(1, 1) > 0 \text{ and } f_{xx}(1, 1) > 0. \text{ Therefore there is a relative minimum at (}x, y\text{) = (}1, 1\text{).} \]

8. \( f(x, y) = x^2 - 6xy - 2y^3 \)

Answer:
\[ f_x(x, y) = 2x - 6y \]
\[ f_y(x, y) = -6x - 6y^2 \]
\[ f_x = f_y = 0 \implies x = 3y \implies -6y^2 = 6(3y) \implies y = 0, -3 \]
\[ y = 0 \implies x = 0 \text{ and } y = -3 \implies x = -9 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = (2)(-12y) - (-6)^2 \]
\[ D(0, 0) < 0. \text{ Therefore there is a saddle point at (}x, y\text{) = (}0, 0\text{).} \]
\[ D(-9, -3) > 0 \text{ and } f_{xx} > 0. \text{ Therefore there is a relative minimum at (}x, y\text{) = }(-9, -3).\]

9. \( f(x, y) = e^{-(x^2+y^2-6y)} \)

Answer:
\[ f_x(x, y) = e^{-(x^2+y^2-6y)}(-2x) \]
\[ f_y(x, y) = e^{-(x^2+y^2-6y)}(-2y + 6) \]
\[ f_x = 0 \implies x = 0 \]
\[ f_y = 0 \implies y = 3 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = e^{-(x^2+y^2-6y)}(4x^2 - 2e^{-(x^2+y^2-6y)})(e^{-(x^2+y^2-6y)}(-2y + 6)^2 - 2ye^{-(x^2+y^2-6y)}) - (e^{-(x^2+y^2-6y)}(-2y + 6)(-2x))^2 \]
\[ D(0, 3) > 0 \text{ and } f_{xx}(0, 3) < 0. \text{ Therefore there is a relative maximum} \]
at \((x, y) = (0, 3)\).

10. \(f(x, y) = (x - 4)\ln(xy)\)
Answer:
\[f_x(x, y) = \ln(xy) + (x - 4)\frac{1}{x}\]
\[f_y(x, y) = (x - 4)\frac{1}{y}\]
\[f_y = 0 \implies x = 4\]
\[x = 4 \implies y = \frac{1}{4}\]
\[D = f_{xx}f_{yy} - (f_{xy})^2\]
\[D = \left(\frac{1}{x} + \frac{4}{x^2}\right)\left(\frac{1}{y^2}\right) - \left(\frac{1}{y}\right)^2\]
\[D(4, \frac{1}{4}) < 0. \text{ Therefore there is a saddle point at } (x, y) = (1, \frac{1}{4}).\]