I. Determine if the demand is elastic, inelastic, or of unit elasticity at the indicated price $p$.

1. $D(p) = -1.3p + 10; \quad p = 4$

2. $D(p) = -1.5p + 25; \quad p = 12$

3. $D(p) = -2p + 40; \quad p = 10$

4. $D(p) = 100 - p^2; \quad p = 10$

II. When an electronics store price a certain brand of stereo at $p$ hundred dollars per set, it is found that $q$ sets will be sold each month, where $q^2 + 2p^2 = 41$.

a. Find the elasticity of demand for the stereos.

b. For a unit price of $p = 4(\$400)$, is the demand elastic, inelastic, or of unit elasticity?

III. Solve these optimization problems.

1. Find two positive numbers whose sum is 100 and whose product is as large as possible.

2. Find two positive numbers $x$ and $y$ whose sum is 30 and are such that $xy^2$ is as large as possible.

3. A store has been selling a popular computer game at the price of $60 per unit, and at this price, players have been buying 50 units per month. The owner of the store wishes to raise the price of the game and estimates that for each $\$2$ increase in price, two fewer units will be sold each month. If each unit cost the store $\$20$, at what price should the game be sold to maximize profit?
4. A city recreation department plans to build a rectangular playground having an area of 49,000 square meters and surround it by a fence. How can this be done using the least amount of fencing?

5. A city recreation department plans to build a rectangular playground having an area of 49,000 square meters and surround it by a fence. The side facing the road cost $10 per meter and the rest of fencing cost $5 a meter. What dimensions of the fence minimize cost?

6. A carpenter has been asked to build an open box with a square base. The sides of the box will cost $4 per square meter, and the base will cost $3 per square meter. What are the dimensions of the box of greatest volume that can be constructed for $96?

7. A closed box with a square base is to have a volume of 300 cubic meters. The material for the top and bottom of the box cost $3 per square meter, and the material for the sides cost $2 per square meter. Can the box be constructed for less than $350?

8. A manufacturer finds that in producing \( x \) units per day \((0 < x < 100)\), three different kinds of cost are involved.
   A. A fixed cost of $1,500 per day in wages.
   B. A production cost of $1.5 per day for each unit produced.
   C. An ordering cost of \( \frac{200}{x^2} \) dollars per day.
   Express the total cost as a function of \( x \) and determine the level of production that results in minimal total cost.
I. Determine if the demand is elastic, inelastic, or of unit elasticity at the indicated price $p$.

1. $D(p) = -1.3p + 10; \quad p = 4$
   
   Answer: 
   
   
   $$E(p) = \frac{\partial q}{\partial p}, \text{ where } q = D(p)$$
   
   $$E(p) = \frac{\partial q}{\partial p} = \frac{p}{-1.3p + 10} D'(p) = \frac{-1.3p + 10}{-1.3p + 10} (-1.3) = -1.3$$
   
   $$|E(4)| = \left| \frac{-1.3(4)}{-1.3(4) + 10} \right| = \frac{5.2}{4.8} > 1$$
   
   Demand is elastic at $p = 4$.

2. $D(p) = -1.5p + 25; \quad p = 12$
   
   Answer: 
   
   $$E(p) = \frac{\partial q}{\partial p}, \text{ where } q = D(p)$$
   
   $$E(p) = \frac{\partial q}{\partial p} = \frac{p}{-1.5p + 25} D'(p) = \frac{-1.5p + 25}{-1.5p + 25} (-1.5) = -1.5$$
   
   $$|E(12)| = \left| \frac{-1.5(12)}{-1.5(12) + 25} \right| = \frac{18}{17} > 1$$
   
   Demand is elastic at $p = 12$.

3. $D(p) = -2p + 40; \quad p = 10$
   
   Answer: 
   
   $$E(p) = \frac{\partial q}{\partial p}, \text{ where } q = D(p)$$
   
   $$E(p) = \frac{\partial q}{\partial p} = \frac{p}{-2p + 40} D'(p) = \frac{-2p + 40}{-2p + 40} (-2) = 2$$
   
   $$|E(10)| = \left| \frac{-2(10)}{-2(10) + 40} \right| = \frac{20}{20} = | -1 | = 1$$
   
   Demand has unit elasticity at $p = 10$. 

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4. \( D(p) = 100 - p^2; \quad p = 12, 1 \)

Answer:
\[
E(p) = \frac{\partial q}{\partial p}, \quad \text{where } q = D(p)
\]
\[
E(p) = \frac{\partial q}{\partial p} = \frac{p}{100-p^2} D'(p) = \frac{p}{100-p^2} (-2p) = -\frac{2p^2}{100-p^2}
\]
\[
|E(12)| = \left| -\frac{2(12)^2}{100-(12)^2} \right| = \left| -\frac{288}{44} \right| = \frac{288}{44} > 1
\]
Demand is elastic at \( p = 12 \)
\[
|E(1)| = \left| -\frac{2(1)^2}{100-(1)^2} \right| = \left| -\frac{2}{99} \right| = \frac{2}{99} < 1
\]
Demand is inelastic at \( p = 1 \)

II. When an electronics store price a certain brand of stereo at \( p \) hundred dollars per set, it is found that \( q \) sets will be sold each month, where \( q^2 + 2p^2 = 41 \).

a. Find the elasticity of demand for the stereos.

Answer:
\[
E(p) = \frac{\partial q}{\partial p}, \quad \text{where } q = D(p)
\]
\[
q^2 + 2p^2 = 41
\]
\[
2q \frac{\partial q}{\partial p} + 4p = 0
\]
\[
2q \frac{\partial q}{\partial p} = -4p
\]
\[
\frac{\partial q}{\partial p} = -\frac{2p}{q}
\]
\[
E(p) = \frac{\partial q}{\partial p} = \frac{p}{q} \frac{\partial q}{\partial p} = \frac{p}{q} \left( -\frac{2p}{q} \right) = -\frac{2p^2}{q^2}
\]
\[
E(p) = -\frac{2p^2}{41-2p^2}
\]

b. For a unit price of \( p = 4(\$400) \), is the demand elastic, inelastic, or of unit elasticity?

Answer:
\[
|E(4)| = \left| -\frac{2(4)^2}{41-2(4)^2} \right| = \left| -\frac{32}{9} \right| = \frac{32}{9} > 1
\]
The demand is elastic at \( p = 4(\$400) \).
III. Solve these optimization problems.

1. Find two positive numbers whose sum is 100 and whose product is a large as possible.
   Answer:
   We did to maximize the product of two numbers, which we will call $x$ and $y$.
   $Product = xy$ and we need to get the equation in terms of one variable.
   We know $x + y = 100$, which means $y = 100 - x$. Which gives us:
   $Product(x) = x(100 - x) = 100x - x^2$
   We want to find the maximum. of $Product(x)$, so first we find the critical points.
   $Product'(x) = 100 - 2x$
   $Product'(x) = 0$ when $x = 50$
   $Product''(x) = -2 < 0$ which means $Product(x)$ is concave down everywhere and therefore there is a maximum at $x = 50$.
   This problem is defined for $x, y > 0$, however since there is only one critical number on this interval we know its the gobal maximum.
   Thus $Product(x)$ has a global maximum at $x = 50$.
   The product is as large as possible is the two numbers are 50 and 50.

2. Find two positive numbers $x$ and $y$ whose sum is 30 and are such that $xy^2$ is as large as possible.
   Answer:
   $Product = xy^2$ and we know that $x + y = 30$.
   $Product(y) = (30 - y)y^2 = 30y - y^3$
   $Product'(y) = 30 - 3y^2$
   $Product'(y) = 0$ when $y = \sqrt{10}, -\sqrt{10}$ $Product''(y) = -6y$ and $Product(\sqrt{10}) < 0$
   There is a relative maximum at $y = \sqrt{10}$ and since it is the only critical number on the given interval($y \geq 0$), there is a absolute maximum at $y = \sqrt{10}$.
   $xy^2$ is as large as possible when $x = 30 - \sqrt{10} \approx 26.8377$ and $y = \sqrt{10} \approx 3.16228$. 
3. A store has been selling a popular computer game at the price of $60 per unit, and at this price, players have been buying 50 units per month. The owner of the store wishes to raise the price of the game and estimates that for each $2 increase in price, two fewer units will be sold each month. If each unit cost the store $20, at what price should the game be sold to maximize profit?

Answer:
We need to maximize profit with respect to price. So we must derive a function for profit that is dependent on price.

Profit = Revenue - Cost

Profit = (games sold)(price of the game) - (games sold)(cost to the store of the game)

We need to derive a function for games sold, \( x \) dependent upon price, \( p \).

We are given that when \( p = 60 \), \( x = 50 \) and \( \frac{\Delta x}{\Delta p} = -\frac{2}{2} = -1 \) for all \( p \).

We have a point \((60, 50)\) and the slope of the function is -1 for \( p \). Thus

\[
(x - 50) = -1(p - 60) \\
x - 50 = -p + 60 \\
x = -p + 110
\]

Profit = (games sold)(price of the game) - (games sold)(cost to the store of the game)

Profit = \( xp - x(cost to the store of the game) \)

Profit = \((-p + 110)p - (-p + 110)20 \)

Profit = \(-p^2 + 110p + p20 - 2200 \)

\( P(p) = -p^2 + 130p - 2200 \)

Now we find the maximum of \( P(p) \).

\( P'(p) = -2p + 130 \)

\( P'(p) = 0 \) when \( p = 65 \)

\( P''(p) = -2 < 0. \) \( P(p) \) is concave down everywhere and there is a maximum at \( p = 65 \).

The games should be sold at a price of $65 to maximize profit.

4. A city recreation department plans to build a rectangular playground having an area of 4,900 square meters and surround it by a fence. How can this be done using the least amount of fencing?

Answer:
We want to minimize the perimeter of the fence.
Perimeter = 2x + 2y (where x and y are the dimensions of the fence.)
We know Area = xy = 4900
Thus P(x) = 2x + 2(4900/x)
P'(x) = 2 − 9800/x^2
P'(x) = 0 when x = 70, −70
P''(x) = 29600/x^3 and P''(70) > 0.
Since this is the only critical number in the range of the interval (x > 0). There is an absolute minimum at x = 70.
The fence could be built with the least amount of fencing if the sides are both 70 meters long.

5. A city recreation department plans to build a rectangular playground having an area of 490,000 square meters and surround it by a fence. The side facing the road cost $10 per meter and the rest of the fencing cost $5 a meter. What dimensions of the fence minimize cost?
Answer:
Cost = 10x + 5x + 5y + 5y (where x is the length of the side of the fence facing the road and y is the length of the adjacent sides.)
We know 490000 = xy
Thus C(x) = 15x + 10(490000/x)
C'(x) = 15 − 4900000/x^2
C'(x) = 0 when x = √4900000/15 = 700√(2/3)
C''(x) = 9800000/x^3 and C''(700√(2/3)) > 0
Since this is the only critical number in the range of the interval (x > 0). There is an absolute minimum at x = 700√(2/3).
The fence should be 700√(2/3) by 700/√(2/3) to minimize cost.

6. A carpenter has been asked to build an open box with a square base. The sides of the box will cost $4 per square meter, and the base will cost $3 per square meter. What are the dimensions of the box of greatest volume that can be constructed for $96?
Answer:
Volume = x^2y (when x is the length of a side of the base and y is the height of the box.)
Cost = 96 = 3x^2 + 4xy + 4xy + 4xy + 4xy = 3x^2 + 16xy
V(x) = x^2((96 - 3x^2)/16x) = 6x - (3/16)x^3
V'(x) = 6 - (9/16)x^2
V'(x) = 0 when \( x = \pm \sqrt{96/9} = \pm (4/3)\sqrt{6} \)
V''(x) = -(9/8)x and V''((4/3)\sqrt{6}) < 0
Since there is only one critical number in the range of the interval \( (x > 0) \). There is an absolute maximum at \( x = (4/3)\sqrt{6} \).
The dimensions of the box of greatest volume that can be constructed for $96 are \((4/3)\sqrt{6}\) by \((4/3)\sqrt{6}\) by 3/8.

7. A closed box with a square base is to have a volume of 300 cubic meters. The material for the top and bottom of the box cost $3 per square meter, and the material for the sides cost $2 per square meter. Can the box be constructed for less than $350?

Answer:
We want to minimize the cost to the box.
Cost = 3x^2 + 3x^2 + 2xy + 2xy + 2xy + 2xy = 6x^2 + 8xy (where \( x \) is length of a side of the base and \( y \) is the length of the height.)
Volume = 300 = x^2y
C(x) = 6x^2 + 8x(300/x^2) = 6x^2 + 2400/x
C'(x) = 12x - 2400/x^2
C'(x) = 0 when \( x = \sqrt[3]{2400/12} = \sqrt[3]{200} \)
C''(\sqrt[3]{200}) > 0. There is absolute minimum at \( x = \sqrt[3]{200} \).
C(\sqrt[3]{200}) = 6\sqrt[3]{200}^2 + 2400/\sqrt[3]{200} \approx 445.482
The box cannot be constructed for less than $350 since the least it can be constructed for is $445.48.

8. A manufacturer finds that in producing \( x \) units per day \( (0 < x < 100) \), three different kinds of cost are involved.
A. A fixed cost of $1,500 per day in wages.
B. A production cost of $1.5 per day for each unit produced.
C. An ordering cost of 200/x^2 dollars per day.
Express the total cost as a function of \( x \) and determine the level of production that results in minimal total cost.
Answer:
Total Cost = 1500 + 1.5x + 200/x^2
TC''(x) = 1.5 - 400/x^3
TC'(x) = 0 when x = \sqrt[3]{400/1.5} = \sqrt[3]{800/3}
TC''(x) = 1200/x^4 > 0 for all x. There is an absolute minimum at
x = \sqrt[3]{800/3}, since it is the only critical point on the interval.
x = \sqrt[3]{800/3} \approx 6.4366
TC(6) = 1514.56 and TC(7) = 1514.58
The total cost will be minimize is the produce 6 units a day.