I. Determine where the graph of the given function is concave upward and concave downward and find all inflection points.

1. \( f(x) = x^2 + 5x + 16 \)
   Answer:
   \[
   f'(x) = 2x + 5 \\
   f''(x) = 2 > 0 \\
   f(x) \text{ is concave up everywhere. There are no inflection points.}
   \]

2. \( f(x) = x^3 + 1 \)
   Answer:
   \[
   f'(x) = 3x^2 \\
   f''(x) = 6x \\
   f''(x) > 0 \text{ for } x > 0 \text{ and } f''(x) < 0 \text{ for } x < 0 \text{ } f(x) \text{ is concave up for } x > 0 \text{ and } f(x) \text{ is concave down for } x < 0 \\
   \text{There is one inflection point at } x = 0
   \]

3. \( f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1 \)
   Answer:
   \[
   f'(x) = 4x^3 + 12x^2 + 12x + 4 \\
   f''(x) = 12x^2 + 24x + 12 = 12(x^2 + 2x + 1) = 12(x + 1)^2 \\
   f''(x) > 0 \text{ for all } x. \\
   f(x) \text{ is concave up for all } x. \\
   \text{There are no inflection points.}
   \]

4. \( f(x) = x(x + 3)^2 \)
   Answer:
   \[
   f'(x) = (x + 3)^2 + x(2x + 3) = (x + 3)(x + 3 + 2x) = (x + 3)(3x + 3) \\
   f''(x) = (3x + 3) + (x + 3)3 = 6x + 12 \\
   f''(x) = 0 \text{ when } x = -2
   \]
\( f''(x) > 0 \) for \( x > -2 \) and \( f''(x) < 0 \) for \( x < -2 \) \( f(x) \) is concave up for \( x > -2 \) and \( f(x) \) is concave down for \( x < -2 \)

There is one inflection point at \( x = -2 \)

5. \( f(x) = (x - 4)^{10/3} \)
   
   Answer:
   
   \[
   \begin{align*}
   f'(x) &= (10/3)(x - 4)^{7/3} \\
   f''(x) &= (70/9)(x - 4)^{4/3} \\
   f''(x) &= 0 \text{ when } x = 4 \\
   f''(x) &> 0 \text{ for } x > 4 \text{ and } f''(x) > 0 \text{ for } x < 4 \text{ } f(x) \text{ is concave up for } x \text{ except for } x = 4. \\
   \text{There are no inflection points.}
   \end{align*}
   \]

6. \( f(x) = 3x^5 - 25x^4 + 11x - 17 \)
   
   Answer:
   
   \[
   \begin{align*}
   f'(x) &= 15x^4 - 100x^3 + 11 \\
   f''(x) &= 60x^3 - 300x^2 = 60x^2(x - 5) \\
   f''(x) &= 0 \text{ when } x = 0, 5 \\
   f''(x) &< 0 \text{ for } x < 0 \text{ and } f''(x) < 0 \text{ for } 0 < x < 5 \text{ and } f''(x) > 0 \text{ for } x > 50 \text{ } f(x) \text{ is concave up for } x > 5 \text{ and } f(x) \text{ is concave down for } x < 0 \text{ and } 0 < x < 5. \\
   \text{There is one inflection point at } x = 5
   \end{align*}
   \]

II. Use the second derivative test to find the relative maxima and minima of the given function.

1. \( f(x) = x^3 + 3x^2 + 1 \)
   
   Answer:
   
   \[
   \begin{align*}
   f'(x) &= 3x^2 + 6x = 2x(x + 2) \\
   f'(x) &= 0 \text{ when } x = 0, -2 \\
   f''(x) &= 6x + 6 \\
   f''(0) &= 6 > 0 \text{ So } f \text{ is concave up at } x = 0 \text{ therefore there is a minimum }
   \end{align*}
   \]
at $x = 0$.

$f''(-2) = -6 < 0$ So $f$ is concave down at $x = -2$ therefore there is a maximum at $x = -2$.

2. $f(x) = x + 1/x$

Answer:

$f'(x) = 1 - 1/x^2$

$f'(x) = 0$ when $x = -1, 1$

$f''(x) = 2/x^3$

$f''(1) = 2 > 0$ So $f$ is concave up at $x = 1$ therefore there is a minimum at $x = 1$.

$f''(-1) = -2 < 0$ So $f$ is concave down at $x = -1$ therefore there is a maximum at $x = -1$.

3. $f(x) = (x + 1)/(x^2 + 1)$

Answer:

$$f'(x) = \frac{(x^2 + 1) - (x + 1)(2x)}{(x^2 + 1)^2} = \frac{-x^2 - 2x + 1}{(x^2 + 1)^2}$$

$f'(x) = 0$ when $x = -1 + \sqrt{2}, -1 - \sqrt{2}$

$$f''(x) = \frac{(-2x-2)(x^2+1)^2 - (-x^2-2)(x^2+1)(2x^2+1)}{(x^2+1)^4}$$

$f''(-1 - \sqrt{2}) > 0$ So $f$ is concave up at $x = -1 - \sqrt{2}$ therefore there is a minimum at $x = -1 - \sqrt{2}$.

$f''(-1 + \sqrt{2}) < 0$ So $f$ is concave down at $x = -1 + \sqrt{2}$ therefore there is a maximum at $x = -1 + \sqrt{2}$.

4. $f(x) = (x/(x + 2))^2$

Answer:
\[ f'(x) = 2 \frac{x}{x+2} \frac{(x+2) - x}{(x+2)^2} = \frac{4x}{(x+2)^3} \]

\[ f'(x) = 0 \text{ when } x = 0 \]

\[ f''(x) = \frac{4(x+2)^3 - 4x3(x+2)^2}{(x+2)^6} \]
\[ = \frac{4(x+2)^2(x+2-3x)}{(x+2)^6} \]
\[ = \frac{4(x+2)^2(-2x+2)}{(x+2)^6} \]
\[ = \frac{-8(x+2)^2(x-1)}{(x+2)^6} \]

\( f''(0) > 0 \) So f is concave up at \( x = 0 \) therefore there is a minimum at \( x = 0 \).

5. \( f(x) = x^2(x-6)^2 \)

Answer:
\[ f'(x) = 2x(x-6)^2 + 2(x-6)x^2 = 2x(x-6)(x-6+x) = 2x(x-6)(2x+6) \]
\[ f'(x) = 0 \text{ when } x = -3, 0, 6 \]

\[ f''(x) = 2(x-6)(2x+6) + 2x(2x+6) + 2x(x-6)(2) \]
\[ f''(-3) > 0 \text{ So f is concave up at } x = -3 \text{ therefore there is a minimum at } x = -3. \]
\[ f''(0) < 0 \text{ So f is concave down at } x = 0 \text{ therefore there is a maximum at } x = 0. \]
\[ f''(6) > 0 \text{ So f is concave up at } x = 6 \text{ therefore there is a minimum at } x = 6. \]

6. \( f(x) = (x+3)^3/(x-1)^2 \)

Answer:
\[ f'(x) = \frac{3(x+3)^2(x-1)^2 - (x+3)^3(x-1)}{(x-1)^4} \]
\[ = \frac{(x+3)^2(x-1)(3(x-1)-2(x+3))}{(x-1)^4} \]
\[ = \frac{(x+3)^2(x-1)(x-9)}{(x-1)^4} \]
\[ f'(x) = 0 \text{ when } x = -3, 9 \]

\[ f''(x) = \frac{2(x + 3)(x - 1)(x - 9) + (x + 3)^2(x - 9) + (x + 3)^2(x - 1)(x - 1)^4 - (x + 3)^2(x - 1)}{(x - 1)^8} \]

\[ f''(-3) = 0 \text{ So } f \text{ has neither a minimum or maximum at } x = -3 \]

\[ f''(9) > 0 \text{ So } f \text{ is concave up at } x = 9 \text{ therefore there is a minimum at } x = 9. \]

III. Sketch a possible graph of \( f(x) \) when \( f'(x) = x(1 - x) \)

IV. An efficiency study of the morning shift (from 8:00 am to 12:00 noon) at a certain factory indicates that an average worker who arrives on the job at 8:00 am will have assembled \( Q(t) = -t^3 + 6t^2 + 15t \) units \( t \) hours later.

a) At what time during the morning is the worker performing most efficiently?

Answer:
We need to find when the rate at which the workers are working is the highest. We need to find the maximum of \( Q'(t) \) when \( 0 \geq t \geq 4 \).

\[ Q'(t) = -3t^2 + 12t + 15 \]

\[ Q''(t) = -6t + 12 \]

\[ Q'''(t) = -6 \]

\[ Q''(t) = 0 \text{ when } t = 2 \text{ and } Q'''(2) = -6 < 0, \text{ therefore } Q'(t) \text{ has a maximum at } t = 2. \]

The workers are performing most efficiently at 10:00 am.
b) At what time during the morning is the worker performing least efficiently?

We know from the last problem there are no local minimums and \( Q'(t) \) is concave down everywhere. That means the workers are increasingly preforming better from 8am to 10 am and from then they are decreasing in productivity. So they are least productive either at 8am or 12am.

\[ Q'(0) = Q'(4) = 15 \]

The workers are performing least efficiently at 8am and noon.

V. A 5-year projection of population trends suggest that \( t \) years from now, the population of a certain community will be \( P(t) = -t^3 + 9t^2 + 48t + 50 \) thousand.

a) At what time during the 5-year period will the population be growing most rapidly?

We need to find when \( P'(t) \) has a maximum.

\[
\begin{align*}
P'(t) &= -3t^2 + 18t + 48 \\
P''(t) &= -6t + 18 \\
P'''(t) &= -6
\end{align*}
\]

\( P''(t) = 0 \) when \( t = 3 \) and \( P'''(3) < 0 \), therefore \( P'(t) \) has a maximum at \( t = 3 \).

The population is growing most rapidly at the start of the 3rd year.

b) At what time during the 5-year period will the population be growing least rapidly?

For the last part, we know there are no local minimums. Therefore the population will be growing least rapidly at an endpoint.

\[ P'(0) = 48 \text{ and } P'(5) = 63. \]

The population will be growing least rapidly at the end of the 5-year period.

c) At what time is the rate of population growth changing most rapidly?

We need to find when \( P''(t) \) has a maximum or a minimum. However we know

\[
\begin{align*}
P''(t) &= -6t + 18 \\
P'''(t) &= -6
\end{align*}
\]

So \( P''(t) \) doesn’t have a relative maximum or maximum since \( P'''(t) \neq 0 \) for any \( t \). We can see from its equation that \( P''(t) \) is decreasing everywhere. So the rate at which the population is growing is growing the most at \( t = 0 \).
VI. An epidemiologist determines that a particular epidemic spreads in such a way that \( t \) weeks after the outbreak, \( N \), hundred new cases will be reported, where

\[
N(t) = \frac{5t}{12 + t^2}
\]

(a) Find \( N'(t) \) and \( N''(t) \).

Answer:

\[
N'(t) = \frac{5(12 + t^2) - 5t(2t)}{(12 + t^2)^2} = \frac{-5t^2 + 60}{(12 + t^2)^2}
\]

\[
N''(t) = -\frac{10t((12 + t^2)^2) - 2(12 + t^2)^2t(-5t^2 + 60)}{(12 + t^2)^4}
\]

\[
= \frac{-2t(12 + t^2)(5(12 + t^2) + 2(-5t^2 + 60)}{(12 + t^2)^4}
\]

(b) At what time is the epidemic at its worst? What is the maximum number of reported new cases?

Answer:

We need when \( N(t) \) has a maximum. \( N'(t) = 0 \) when \( t = -\sqrt{12}, \sqrt{12} \) and \( N''(-\sqrt{12}) > 0 \) and \( N''(\sqrt{12}) < 0 \). Therefore \( N(t) \) has a maximum at \( t = \sqrt{12} \). \( N(\sqrt{12}) = 2.5\sqrt{12} \).

So the epidemic is that worst after 3.46 weeks. and there are about 866 new cases.

c) Health officials declare the epidemic to be under control when the rate of reported new cases is minimized. When does this occur? What number of new cases will be reported at that time?

Answer:

We need to find when \( N'(t) \) has a minimum. We know

\[
N''(t) = \frac{-2t(12 + t^2)(5(12 + t^2) + 2(-5t^2 + 60)}{(12 + t^2)^4}
\]

\[
= \frac{-2t(12 + t^2)(-5t^2 + 180)}{(12 + t^2)^4}
\]

\( N''(t) = 0 \) when \( t = 6 \)

From the first derivative test we see that \( N'(t) \) has a minimum at \( t = 6 \).
\[ N(6) = \frac{30}{48} = \frac{5}{8} \]
The epidemic is under control after 6 weeks and there about 62 new cases.