I. Exponential growth and decay
A. The half-life of fakeium-20 is 40 years. Suppose we have a 100-mg sample.

1. Find the mass that remains after \( t \) years.

2. How much of the sample remains after 100 years?

3. After how long until only 6.25 mg remain?

B. A curve passes through the point (0,6) and has the property that the slope of the curve at every point \( P \) is 3 times the \( y \)-coordinate of \( P \). What is the equation of the curve?

II. Related Rates

1. The radius of a sphere is increasing at a rate of 5 mm/s. How fast is the volume increasing when the diameter is 20 mm?

2. A particle is moving along hyperbola \( x^2y = 8 \). As it reaches the point (2,2), the \( y \)-coordinate is decreasing at a rate of 3 cm/s. How fast is the \( x \)-coordinate of the point changing at that instant?

3. A particle moves along the curve \( y = 3 \sin(\pi x/3) \). As the particle passes through the point (3,0), its \( x \)-coordinate increases at a rate of 2 cm/s. How fast is the distance from the particle to the origin changing at this instant?

4. A trough is 20 ft long and its ends have the shape of isosceles triangles that are 4 ft across at the top and have a height of 2 ft. If the trough is being filled with water at a rate of 12 ft\(^3\)/min, how fast is the water level rising when the water is 6 inches deep?
5. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

6. Two people start from the same point. One walks east at 3 mi/h and the other walks north at 4 mi/h. How fast is the distance between the people changing after 15 minutes?

7. The top of a ladder slides down a vertical wall at a rate of 1 m/s. At the moment when the bottom of the ladder is 4 m from the wall, it slides away from the wall at a rate of 0.75 m/s. How long is the ladder?
I. Exponential growth and decay
A. The half-life of fakeium-20 is 40 years. Suppose we have a 100-mg sample.

1. Find the mass that remains after \( t \) years.
   Answer:
   We know:
   \[
   y = y_0 e^{kt}
   \]

   \[y_0 = 100\] The initial mass.
   The half-life of 40 years implies \( 50 = 100e^{k\cdot40} \)

   \[
   \begin{align*}
   50 & = 100e^{40k} \\
   \frac{1}{2} & = e^{40k} \\
   \ln(0.5) & = 40k \\
   \ln(0.5) & = k
   \end{align*}
   \]

   \[
   y = 100e^{-\ln(2)\frac{t}{40}}
   \]

2. How much of the sample remains after 100 years?
   Answer:
   \[
   y = 100e^{-\ln(2)\frac{100}{40}} = 100e^{-\ln(2)\cdot\frac{5}{2}} = 100e^{\ln(2^{-\frac{5}{2}})} = 100(2^{-\frac{5}{2}}) = \frac{25}{\sqrt{2}}
   \]

3. After how long until only 6.25 mg remain?
   Answer:
\[
\frac{6.25}{100} = e^{-\frac{\ln(2)}{40} t}
\]
\[
\frac{1}{16} = e^{-\frac{\ln(2)}{40} t}
\]
\[
\ln(1/16) = -\frac{\ln(2)}{40} t
\]
\[
-4 \ln(2) = - \ln(2) \frac{t}{40}
\]
\[
4 = \frac{t}{40}
\]
\[
160 = t
\]

It will take 160 years.

B. A curve passes through the point (0,6) and has the property that the slope of the curve at every point \( P \) is 3 times the \( y \)-coordinate of \( P \). What is the equation of the curve?

Answer:
We’re given \( y’ = 3y \) which gives us \( y = y_0 e^{3x} \)
\[
6 = y_0 e^0 = y_0
\]
\[
y = 6e^{3x}
\]

II. Related Rates

1. The radius of a sphere is increasing at a rate of 5 mm/s. How fast is the volume increasing when the diameter is 20 mm?

Answer:
The volume of a sphere is \( V = \frac{4}{3} \pi r^3 \)
\[
\frac{dV}{dt} = \frac{d}{dt} \frac{4}{3} \pi r^3 \\
\frac{dV}{dt} = \frac{4\pi}{3} 3r^2 \frac{dr}{dt} \\
\frac{dV}{dt} = 4\pi (10)^2 5 \\
\frac{dV}{dt} = 20000\pi
\]

2. A particle is moving along hyperbola \( x^2 y = 8 \). As it reaches the point \((2,2)\), the \( y \)-coordinate is decreasing at a rate of 3 cm/s. How fast is the \( x \)-coordinate of the point changing at that instant?

Answer:
We’re given: \( \frac{dy}{dt} = -3 \)
We want: \( \frac{dx}{dt} \) when \((x,y) = (2,2)\)

\[
x^2 y = 8 \\
(x^2 y)' = (8)' \\
2x \frac{dx}{dt} y + x^2 \frac{dy}{dt} = 0
\]

Plug in what we know
\[
2(2) \frac{dx}{dt} 2 + 2^2 (-3) = 0 \\
8 \frac{dx}{dt} = 12 \\
\frac{dx}{dt} = \frac{3}{2}
\]

The \( x \)-coordinate is changing at \( \frac{3}{2} \) cm/s.

3. A particle moves along the curve \( y = 3 \sin(\pi x/3) \). As the particle passes through the point \((3,0)\), its \( x \)-coordinate increases at a rate of 2
cm/s. How fast is the distance from the particle to the origin changing at this instant?
Answer:
We’re given: \( \frac{dx}{dt} = 2 \)
We want: \( \frac{dD}{dt} \) where D is the distance between (x,y) and the origin, when \((x, y) = (3, 0)\).
We use the distance formula:

\[
D^2 = x^2 + y^2
\]
\[
D^2 = x^2 + (3 \sin(\pi x / 3))^2
\]
\[
\frac{d}{dt} D^2 = \frac{d}{dt} x^2 + 9 \sin^2(\pi x / 3)
\]
\[
2d\frac{dD}{dt} = 2x \frac{dx}{dt} + 18 \sin(\pi x / 3) \cos(\pi x / 3) \frac{\pi dx}{3\ dt}
\]
\[
\frac{d}{dt} D = x \frac{dx}{dt} + 9 \sin(\pi x / 3) \cos(\pi x / 3) \frac{\pi dx}{3\ dt}
\]
Plug in what we know

\[
(3) \frac{dD}{dt} = (3)(2) + 9 \sin(2\pi / 3) \cos(2\pi / 3) \frac{\pi}{3}(2)
\]
\[
(3) \frac{dD}{dt} = 6 + 6 \sqrt{3} - 1 \pi
\]
\[
(3) \frac{dD}{dt} = 6 - \frac{3}{2} \sqrt{3} \pi
\]
\[
\frac{dD}{dt} = 2 - \frac{\sqrt{3} \pi}{2}
\]

4. A trough is 20 ft long and its ends have the shape of isosceles triangles that are 4 ft across at the top and have a height of 2 ft. If the trough is being filled with water at a rate of 12 ft\(^3\)/min, how fast is the water level rising when the water is 6 inches deep?
Answer:
We’re given: \( \frac{dV}{dt} = 12 \), where V is Volume of water
We want: \( \frac{dH}{dt} \), when H is the height of the water when H = .5 feet.
The volume of the water is \( V = 20 \left( \frac{H \cdot B}{2} \right) \), where H is the height of the water, and B is the length of water parallel to the base of the triangle.
Using similar triangles we see that

\[ \frac{B}{H} = \frac{4}{2} \implies B = 2H \]

So we get:

\[ V = 20 \frac{HB}{2} \]
\[ V = 10H(2H) \]
\[ V = 20H^2 \]

\[ \frac{d}{dt}V = \frac{d}{dt}20H^2 \]
\[ \frac{dV}{dt} = 40H \frac{dH}{dt} \]

Plug in what we know

\[ 12 = 40(0.5) \frac{dH}{dt} \]
\[ 12 = 20 \frac{dH}{dt} \]
\[ \frac{3}{5} = \frac{dH}{dt} \]

The water is rising .6 ft/s.

5. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

Answer:

We’re given: \( \frac{dx}{dt} = 8 \), where \( x \) is the distance along the ground.

We want: \( \frac{d\theta}{dt} \), where \( \theta \) is the angle between the string and the horizontal.

An equation that relates \( x \) and \( \theta \) is: \( \cos(\theta) = \frac{x}{200} \)
\[
\cot(\theta) = \frac{x}{100}
\]
\[
\frac{d}{dt} \cos(\theta) = \frac{d}{dt} \frac{x}{100}
\]
\[
-\csc^2(\theta) \frac{d\theta}{dt} = \frac{1}{100} \frac{dx}{dt}
\]
Plug in what we know
\[
-\csc^2(\cos^{-1}(1/2)) \frac{d\theta}{dt} = \frac{1}{100} \frac{8}{100}
\]
\[
-2 \frac{d\theta}{dt} = \frac{2}{25}
\]
\[
-4 \frac{d\theta}{dt} = \frac{2}{25}
\]
\[
\frac{d\theta}{dt} = \frac{1}{50}
\]
The angle is decreasing at \( \frac{1}{50} \) rad/sec.

6. Two people start from the same point. One walks east at 3 mi/h and the other walks north at 4 mi/h. How fast is the distance between the people changing after 15 minutes? Answer:

We’re given: \( \frac{dx}{dt} = 3 \) and \( \frac{dy}{dt} = 4 \), where \( x \) is the distance traveled east and \( y \) is the distance traveled north.

We want: \( \frac{dD}{dt} \), where \( D \) is the distance between the people.

We use the formula:
\[
D^2 = x^2 + y^2
\]
\[ D^2 = x^2 + y^2 \]
\[ \frac{d}{dt}D^2 = \frac{d}{dt}x^2 + y^2 \]
\[ 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \]
\[ \frac{dD}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \]

Plug in what we know
\[ 5\frac{dD}{dt} = 3(0.25)(3) + (4)(0.25)4 \]
\[ 5\frac{dD}{dt} = 2.25 + 4 \]
\[ 5\frac{dD}{dt} = 6.25 \]
\[ \frac{dD}{dt} = 1.25 \]

The distance is changing at a rate of 1.25 ft/s.

7. The top of a ladder slides down a vertical wall at a rate of 1 m/s. At the moment when the bottom of the ladder is 4 m from the wall, it slides away from the wall at a rate of 0.75 m/s. How long is the ladder?

Answer:
We’re given: \( \frac{dx}{dt} = 1 \) and \( \frac{dy}{dt} = 0.75 \), where \( x \) is distance along the wall and \( y \) is the distance along the floor.
We want: \( L \) when \( y = 4 \) and \( L \) is the length of the ladder.
We use the formula: \( L^2 = x^2 + y^2 \)
\[ L^2 = x^2 + y^2 \]
\[ \frac{d}{dt} L^2 = \frac{d}{dt} (x^2 + y^2) \]
\[ 2L \frac{dL}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \]
\[ \frac{dL}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \]

Plug in what we know

\[ L(0) = x(-1) + 4(.75) \]
\[ 0 = -x + 3 \]
\[ x = 3 \]

\[ L^2 = 3^2 + 4^2 = 25 \implies L = 5 \]

The ladder is 5 feet long.