Chapter 13: Fair Division

October 11, 2013
Cake-Division Procedures
Vickrey Auctions
Motivating Question

How can divide a piece of cake between 3 people?
Cake-Division Procedure

A **cake-division procedure** for *n* players is a procedure that the players can use to allocate a cake among themselves so that each player has a strategy that will guarantee that player a piece with which he or she is “satisfied”, even in the face of collusion by the others.

Proportional Procedure

A cake-division procedure (for *n* players) will be called **proportional** if each player’s strategy guarantees that player a piece of size or value at least $1/n$ of the whole in his or her own estimation.
Steinhaus Proportional Procedure

Assume there are 3 players: Bob, Carol and Ted.

1. Bob will cut the cake into 3 pieces (what he believes are 3 equal) pieces.

2. Carol will approve a piece. Ted will approve a piece. If they are different pieces, each take their piece and Bob gets the remaining piece. Done.

3. If they approve the same piece. Then Bob gets one of disapprove pieces. The other two pieces are put together and Carol and Ted use divide and choose on them. Done.
Banach-Knaster Proportional Procedure

Assume there are 4 players: Bob, Carol, Ted and Alice.

1. Bob cuts a piece that he thinks is size 1/4.
2. If Carol thinks its at most 1/4 size, she will give it to Ted. If she thinks its too big she trim the excess of then give it Ted.
3. Ted does the same thing and then passes it to Alice. Alice will trim off any excess.
4. The last person to trim the cake gets the piece.
5. The process starts over at step with the remaining cake.
6. The remaining two people will use divide and choose on the remaining cake.
The Selfridge-Conway Envy-Free Procedure for Three Players

1. Player 1 cuts the cake into three pieces that he considers to be the same size. He hands the three pieces to Player 2.
2. Player 2 trims at most one of the three pieces to create at least a two-way tie for largest. Setting the trimming aside, Player 2 hands the three pieces to Player 3.
3. Player 3 now chooses, from among the three pieces, one that he considers to be at least tied for largest.
4. Player 2 next chooses, from the two remaining pieces, one that she considers to be at least tied for largest, with the provision that if she trimmed a piece in Step 2, and player 3 did not choose this piece, then she must choose it.
5. Player 1 receives the remaining piece.
Motivating Question

If you selling an item? on E-bay who should get the item?
Vickrey Auctions

A Vickrey Auction

In a Vickrey auction, bidders independently submit sealed bids for the object begin sold. The winner is the high bidder, but he or she pays only the amount of the second-highest bid.

Strategy for Bidding in a Vickrey Auction

In a Vickrey auction, a bidder can never do better than that achieved by a bid of exactly what the object is worth to that bidder.
Last Time

- Fair division and Organ Transplant Policies
- Taking turns
- Divide and Choose
Problem

Which of the four recipients should get the kidney with the following characteristics:

<table>
<thead>
<tr>
<th>Potential Recipient</th>
<th>Months Waiting</th>
<th>Antigens Matched</th>
<th>Percent Sensitized</th>
<th>Potential Recipient</th>
<th>Months Waiting</th>
<th>Antigens Matched</th>
<th>Percent Sensitized</th>
<th>Points Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9</td>
<td>2</td>
<td>20</td>
<td>A</td>
<td>10</td>
<td>4</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>B</td>
<td>7.5</td>
<td>6</td>
<td>0</td>
<td>13.5</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>4</td>
<td>40</td>
<td>C</td>
<td>5</td>
<td>8</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>6</td>
<td>60</td>
<td>D</td>
<td>2.5</td>
<td>12</td>
<td>6</td>
<td>20.5</td>
</tr>
</tbody>
</table>

So D gets the kidney
Mark and Fred have inherited a number of items from their parent’s estate, with no indication of who gets what. They rank the items from most preferred to least preferred as follows:

<table>
<thead>
<tr>
<th>Mark</th>
<th>Fred</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truck</td>
<td>Boat</td>
</tr>
<tr>
<td>Tractor</td>
<td>Tractor</td>
</tr>
<tr>
<td>Boat</td>
<td>Car</td>
</tr>
<tr>
<td>Car</td>
<td>Truck</td>
</tr>
<tr>
<td>Tools</td>
<td>Motorcycle</td>
</tr>
<tr>
<td>Motorcycle</td>
<td>Tools</td>
</tr>
</tbody>
</table>

Assume that Mark and Fred use the bottom-up strategy and that Mark gets to choose first. Determine Mark’s first choice and the final allocation.

**Bottom-up Strategy:**

Mark: Tractor, Truck, Tools

Fred: Boat, Car, Motorcycle
Adjusted Winner Procedure

Basic Steps in the Adjusted Winner Procedure

**Step 0:** Each party distributes 100 points over the items in a way that reflects each item’s relative worth to that party.

**Step 1:** Each item on which the assigned points differ is initially given to the party that assigned it more points. Add up the total number of points each party feels that he or she has received. The party with the fewest points is now given all the items on which both parties placed the same number of points. Once again, add up the total number of points each party feels that he or she has received. The party with the most points is called the *initial winner*; the other party is called the *initial loser*.

**Step 2:** For each item given to the initial winner, calculate the “point ratio”
Step 3: Start moving items from the initial winner to the initial loser in ascending order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be split or shared and is thus called the **shared item**.

Step 4: Let $x$ represent the fractional part of the shared item that will be moved from the initial winner to the initial loser. Write a formula that equates each party’s total points after the sharing of this item.

Step 5: Solve the equation and state the final division of items between the two parties.
Suppose that Calvin and Hobbes discover a sunken pirate ship and must divide their loot. How should they divide their loot using the adjusted winner procedure.

<table>
<thead>
<tr>
<th>Object</th>
<th>Calvin’s Points</th>
<th>Hobbes’s Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cannon</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Anchor</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Unopened Chest</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Doubloon</td>
<td>11</td>
<td>14</td>
</tr>
<tr>
<td>Figurehead</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>Sword</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>Cannon ball</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Wooden leg</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Flag</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Crow’s nest</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Calvin is initially awarded the Cannon (10), Sword (15), Cannon Ball (5), Wooden Leg (2), Flag (10), Crow’s nest (2).
Hobbes is initially awarded the Anchor (20), Unopened Chest (20), Doubloon (14), Figurehead (30).
Calvin’s points, \(10 + 15 + 5 + 2 + 10 + 2 = 44\) Initial Loser
Hobbes’s points, \(20 + 20 + 14 + 30 = 84\) Initial Winner
Point ratio
Anchor 20/10 = 2
Unopened Chest 20/15 = 1.333
Doubloon 14/11 = 1.2727
Figurehead 30/20 = 1.5
The Doubloon is then transferred to Calvin and the points are now:
Calvin’s points, 55
Hobbes’s points, 70
The Unopened Chest is transferred next, however if it is transferred
the points will be.
Calvin’s points, 70
Hobbes’s points, 50
Which will switch the initial winner and initial loser, so the item
must be shared.
So we must setup an equation to split the item

\[ 55 + 15x = 70 - 20x \]

\[ 35x = 15 \]
\[ x = \frac{3}{7} \]
\[ x = 0.428571429 \]
Knaster’s Inheritance procedure

Basic Steps in Knaster’s Inheritance Procedure with $n$ Heirs

For each object, the following steps are performed:

**Step 1:** The heirs— independently and simultaneously— submit monetary bids for the object.

**Step 2:** The high bidder is awarded the object, and he or she places all but $1/n$ of his or her bid in a kitty. So, if there are four heirs ($n=4$), then he or she places all but one-fourth— that is, three-fourths— of his or here bid in a kitty.

**Step 3:** Each of the other heirs withdraws from the kitty $1/n$ of his or her bid.

**Step 4:** The money remaining in the kitty is divided equally among the $n$ heirs.
John and Mary inherit their parent’s old house and classic car. John bids $28,225 on the car and $55,900 on the house. Mary bids $32,100 on the car and $59,100 on the house. How should they arrive at a fair division?
Mary gets the car and the house. She places $\frac{32100}{2} + \frac{59100}{2}$ in the kitty. Then John withdraws $\frac{28225}{2} + \frac{55900}{2}$ from the kitty. They split the remainder. The net effect is Mary gets the car and the house and pays John $43,831.25.
Use adjusted winner procedure to resolve Mike and Phil’s roommate issues.

<table>
<thead>
<tr>
<th>Issue</th>
<th>Mike’s Points</th>
<th>Phil’s Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stereo Level</td>
<td>4</td>
<td>22</td>
</tr>
<tr>
<td>Smoking rights</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Room party policy</td>
<td>50</td>
<td>25</td>
</tr>
<tr>
<td>Cleanliness</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Alcohol use</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Phone time</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Lights out time</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Visitor policy</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Issue</td>
<td>Mike’s Points</td>
<td>Phil’s Points</td>
</tr>
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<tr>
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<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Lights out time</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Visitor policy</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Mike gets Room party policy, Cleanliness, Lights out time, $50 + 6 + 10 = 66$

Phil gets Stereo Level, Smoking rights, Phone time, Visitor policy, $22 + 20 + 8 + 5 = 55$

So Phil also gets Alcohol use which gives him 70 points making him the initial winner.
Problem Answer

Point ratio
Stereo Level 22/4
Smoking rights 20/10
Phone time 8/1
Visitor policy 5/4
Alcohol use 15/15
So they split the Alcohol use

\[66 + 15(x) = 70 - 15x\]
\[30x = 4\]
\[x = 2/15\]

So Mike gets Room party policy, Cleanliness, Lights out time, (2/15) Alcohol use
Phil gets Stereo Level, Smoking rights, Phone time, Visitor policy, (13/15) Alcohol use
Next time

- Chapter 21, Savings Models