

Chapter 13: Fair Division

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Motivating Question

In 1991 Ivana and Donald Trump divorce. The marital assets included a 45-room mansion in Greenwich, Connecticut; the 118-room Mar-a-Lago mansion in Palm Beach, Florida; an apartment in the Trump Plaza; a 50-room Trump Tower triplex; and just over \$1 million in cash and jewelry. How should assets be divided up?

Desirable properties of a fair division

Equitable

A fair-division procedure is said to be **equitable** if each player believes he or she received the same fractional part of the total volume.

Envy-Free

A fair-division procedure is said to be **envy-free** if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large (or at least as desirable) as that received by any other player, no matter what the other players do.

Desirable properties of a fair division

Pareto-Optimal

A fair-division procedure is said to be **Pareto-optimal** if it produces an allocation of the property such that no other allocation achieved by any means whatsoever can make any one player better off without making some other player worse off.

Adjusted Winner Procedure

Basic Steps in the Adjusted Winner Procedure

- Step 0:** Each party distributes 100 points over the items in a way that reflects each item's relative worth to that party.
- Step 1:** Each item on which the assigned points differ is initially given to the party that assigned it more points. Add up the total number of points each party feels that he or she has received. The party with the fewest points is now given all the items on which both parties placed the same number of points. Once again, add up the total number of points each party feels that he or she has received. The party with the most points is called the **initial winner**; the other party is called the **initial loser**.
- Step 2:** For each item given to the initial winner, calculate the **“point ratio”**

Adjusted Winner Procedure

Basic Steps in the Adjusted Winner Procedure Cont.

- Step 3:** Start moving items from the initial winner to the initial loser in ascending order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be split or shared and is thus called the **shared item**.
- Step 4:** Let x represent the fractional part of the shared item that will be moved from the initial winner to the initial loser. Write a formula that equates each party's total points after the sharing of this item.
- Step 5:** Solve the equation and state the final division of items between the two parties.

Adjusted Winner Procedure Example

Step 0: Each party distributes 100 points over the items in a way that reflects each item's relative worth to that party.

Martial Assets	Donald's Points	Ivana's Points
Connecticut estate	10	38
Palm Beach mansion	40	20
Trump Plaza apartment	10	30
Trump Tower triplex	38	10
Cash and jewelry	2	2

Adjusted Winner Procedure Example

Step 1: Each item on which the assigned points differ is initially given to the party that assigned it more points. Add up the total number of points each party feels that he or she has received. The party with the fewest points is now given all the items on which both parties placed the same number of points. Once again, add up the total number of points each party feels that he or she has received. The party with the most points is called the **initial winner**; the other party is called the **initial loser**.

Donald gets: Palm Beach mansion, Trump Tower triplex,
 $40 + 38 = 78$ **initial winner**

Ivana gets: Connecticut estate, Trump Plaza apartment, Cash and jewelry
 $38 + 30 + 2 = 70$, **initial loser**

Adjusted Winner Procedure Example

Step 2: For each item given to the initial winner, calculate the **“point ratio”**

Palm Beach mansion has point ratio $40/20 = 2$

Trump Tower triplex has point ratio $38/10 = 3.8$

Adjusted Winner Procedure Example

Step 3: Start moving items from the initial winner to the initial loser in ascending order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be split or shared and is thus called the **shared item**.

The first item to be transferred is the Palm Beach mansion. However we can't give the entire mansion because it will make Donald have less points so we have to split it.

Adjusted Winner Procedure Example

Step 4: Let x represent the fractional part of the shared item that will be moved from the initial winner to the initial loser. Write a formula that equates each party's total points after the sharing of this item.

To equalize point totals we have solve for x where,
 $78 - 40x = 70 + 20x.$

Adjusted Winner Procedure Example

Step 5: Solve the equation and state the final division of items between the two parties.

$$78 - 40x = 70 + 20x$$

$$8 = 60x$$

$$x = 2/15$$

Adjusted Winner Procedure Example

So Donald will get $13/15$ ownership of the Palm Beach Mansion for a total of about 72.7 of his points, and Ivana get the rest (for about 72.7 of her points)

Donald gets: Trump Tower triplex and $13/15$ of the Palm Beach Mansion.

Ivana gets: Connecticut estate, Trump Plaza apartment, Cash and jewelry and $2/15$ of the Palm Beach Mansion.

Adjusted Winner Procedure

The Adjusted Winner Procedure:

- Equitable
- Envy-Free
- Pareto-Optimal

Motivating Problem

Suppose a woman died and left her house to her four children, Bob, Carol, Ted and Alice.
Who should get the house?

Knaster's Inheritance procedure

Basic Steps in Knaster's Inheritance Procedure with n Heirs

For each object, the following steps are performed:

- Step 1:** The heirs- independently and simultaneously- submit monetary bids for the object.
- Step 2:** The high bidder is awarded the object, and he or she places all but $1/n$ of his or her bid in a kitty. So, if there are four heirs ($n=4$), then he or she places all but one-fourth- that is, three-fourths-of his or here bid in a kitty.
- Step 3:** Each of the other heirs withdraws from the kitty $1/n$ of his or her bid.
- Step 4:** The money remaining in the kitty is divided equally among the n heirs.

Knaster's Inheritance procedure Example

The bids:

Bob	Carol	Ted	Alice
\$120,000	\$200,000	\$140,000	\$180,000

Carol wins the bid, so she must put $(200,000) \cdot (3/4)$ in the kitty.

Bob then withdraws $(120,000) \cdot (1/4)$ from the kitty.

Ted then withdraws $(140,000) \cdot (1/4)$ from the kitty.

Alice then withdraws $(180,000) \cdot (1/4)$ from the kitty.

So there is $150,000 - 30,000 - 35,000 - 45,000 = 40,000$ dollars left in the kitty.

So each person gets an additional \$10,000 dollars.

In the end the get:

Bob	Carol	Ted	Alice
\$40,000	House -140,00	\$45,000	\$55,000

Knaster's Inheritance procedure Continue

What do you do if there is more than one item?

	Bob	Carol	Ted	Alice
House	\$120,000	\$200,000	\$140,000	\$180,000
Cabin	\$60,000	\$40,000	\$90,000	\$50,000
Boat	\$30,000	\$24,000	\$20,000	\$20,000

Knaster's Inheritance procedure Continue

	Bob	Carol	Ted	Alice
Cabin	\$60,000	\$40,000	\$90,000	\$50,000

The result

	Bob	Carol	Ted	Alice
	\$22,500	\$17,500	Cabin-\$60,000	\$20,000

Knaster's Inheritance procedure Continue

	Bob	Carol	Ted	Alice
Boat	\$30,000	\$24,000	\$20,000	\$20,000

The result

	Bob	Carol	Ted	Alice
Boat	-\$20,875	\$7625	\$6625	\$6625

Knaster's Inheritance procedure Continue

Final result:

Bob: Boat + \$41,625

Carol: House - \$114,875

Ted: Cabin - \$8375

Alice: \$81,625

Problem

Suppose that Calvin and Hobbes discover a sunken pirate ship and must divide their loot. How should they divide their loot using the adjusted winner procedure.

Object	Calvin's Points	Hobbes's Points
Cannon	10	5
Anchor	10	20
Unopened Chest	15	20
Doubloon	11	14
Figurehead	20	30
Sword	15	6
Cannon ball	5	1
Wooden leg	2	1
Flag	10	2
Crow's nest	2	1

Problem

John and Mary inherit their parent's old house and classic car. John bids \$28,225 on the car and \$55,900 on the house. Mary bids \$32,100 on the car and \$59,100 on the house. How should they arrive at a fair division?

- Fair division and Organ Transplant Policies
- Taking Turns