CONTINUITY, LIMITS AT INFINITY

Instructions. Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

1. \( \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \)

We have

\[
\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{x\sqrt{2 + \frac{1}{x^2}}}{x(3 - \frac{5}{x})} = \lim_{x \to -\infty} \frac{-x\sqrt{2 + \frac{1}{x^2}}}{x(3 - \frac{5}{x})} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \frac{1}{x^2}}}{3 - 0} = -\frac{\sqrt{2}}{3}.
\]

NOTE: A common mistake in calculating this limit is incorrectly writing \( \sqrt{x^2} = x \). This is not true if \( x < 0 \). Since we are taking the limit as \( x \) approaches \(-\infty\), (i.e. \( x \) is becoming larger and larger in the negative direction), we have \( \sqrt{x^2} = -x \) in this case.

2. \( \lim_{x \to \infty} \left( \sqrt{x^2 + 1} - x \right) \)

We have

\[
\lim_{x \to \infty} \sqrt{x^2 + 1} - x = \lim_{x \to \infty} \left( \sqrt{x^2 + 1} - x \right) \left( \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x} \right) = \lim_{x \to \infty} \frac{\left( \sqrt{x^2 + 1} \right)^2 - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x} = \frac{1}{\infty} = 0.
\]

NOTE: It is tempting to avoid these calculations and instead say as \( x \) approaches infinity that both \( \sqrt{x^2 + 1} \) and \( x \) approach \( \infty \) as well; hence, the limit is \( \infty - \infty = 0 \). But this is incorrect. In fact, \( \infty - \infty \neq 0 \). To see why, consider the following limit \( \lim_{x \to \infty} (x - (x + 1)) \). Here again, it is tempting to reason that \( x \) and \( x + 1 \) approach \( \infty \) as \( x \to \infty \) and thus the limit is \( \infty - \infty = 0 \), but upon seeing that \( x - (x + 1) = -1 \), the correct answer for the limit is \(-1\).

3. \( \lim_{x \to +\infty} \frac{x^2 + x}{3 - x} \)
We have
\[
\lim_{x \to +\infty} \frac{x^2 + x}{3 - x} = \lim_{x \to +\infty} \frac{x(x + 1)}{(\frac{3}{x} - 1)}
\]
\[
= \lim_{x \to +\infty} \frac{x + 1}{\frac{3}{x} - 1}
\]
\[
= +\infty \quad \text{or} \quad -\infty.
\]

4. Determine the equation for each horizontal asymptote on the graph of the following function. Your answer must be justified with limits.

\[
f(x) = \frac{36 + 42e^{6x}}{2e^{6x} - 46}
\]

Recall that \(\lim_{x \to -\infty} e^x = 0\) or equivalently \(\lim_{x \to \infty} e^{-x} = 0\). With this, we have
\[
\lim_{x \to -\infty} \frac{36 + 42e^{6x}}{2e^{6x} - 46} = \frac{36 + 0}{0 - 46} = \frac{36}{-46} = \frac{18}{23},
\]
and
\[
\lim_{x \to \infty} \frac{36 + 42e^{6x}}{2e^{6x} - 46} = \lim_{x \to \infty} \frac{e^{6x} (36e^{-6x} + 42)}{e^{6x} (2 - 46e^{-6x})}
\]
\[
= \lim_{x \to \infty} \frac{36e^{-6x} + 42}{2 - 46e^{-6x}} = \frac{0 + 42}{2 - 0} = \frac{42}{2} = 21.
\]

The horizontal asymptotes are \(y = \frac{-18}{23}\) and \(y = 21\).
5. Find the values of $a$ and $b$ that make $f$ continuous everywhere

$$ f(x) = \begin{cases} 
\frac{x^2-4}{x-2} & \text{if } x < 2 \\
ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\
2x - a + b & \text{if } x \geq 3 
\end{cases} $$

Recall that a function $f(x)$ is continuous at the point $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Further, recall that $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$. We will use this to find $a$ and $b$. On one hand, we must have

$$ \lim_{x \to 2^{-}} f(x) = \lim_{x \to 2^{+}} f(x) $$

$$ = \lim_{x \to 2^{-}} \frac{x^2 - 4}{x - 2} $$

$$ = \lim_{x \to 2^{-}} \frac{(x - 2)(x + 2)}{x - 2} $$

$$ = \lim_{x \to 2^{-}} x + 2 = 4. $$

On the other hand, we must have

$$ \lim_{x \to 2^{+}} f(x) = \lim_{x \to 2^{-}} f(x) $$

$$ = \lim_{x \to 2^{+}} ax^2 - bx + 3 $$

$$ = a(2)^2 - b(2) + 3 = 4a - 2b + 3. $$

So equating the left and right hand limits, we get that $4 = 4a - 2b + 3$, or equivalently $1 = 4a - 2b$. In similar fashion, we observe $\lim_{x \to 3} f(x)$. On one hand, we must have

$$ \lim_{x \to 3^{-}} f(x) = \lim_{x \to 3^{+}} f(x) $$

$$ = \lim_{x \to 3^{-}} ax^2 - bx + 3 $$

$$ = a(3^2) - b(3) + 3 = 9a - 3b + 3. $$

On the other hand, we must have

$$ \lim_{x \to 3^{+}} f(x) = \lim_{x \to 3^{-}} f(x) $$

$$ = \lim_{x \to 3^{+}} 2x - a + b $$

$$ = 2(3) - a + b = 6 - a + b. $$

Thus, equating these left and right hand side limits, we get that $9a - 3b + 3 = 6 - a + b$, or equivalently, $10a - 4b = 3$. Solving the system of equations

$$ 4a - 2b = 1 $$

$$ 10a - 4b = 3, $$

yields $a = b = \frac{1}{2}$.

6. Fill in the blanks: The Intermediate Value Theorem says that

IF

$f(x)$ is continuous on the interval $[a,b]$ and $c$ is any value in the interval $[f(a), f(b)]$,

THEN There is an $x_0$ in the interval $[a,b]$ with $f(x_0) = c$. 

7. Use the Squeeze Theorem to find \( \lim_{x \to +\infty} \frac{\cos(x^2 \sin(x))}{e^x} \)

Recall that \(-1 \leq \cos(x) \leq 1\) for any real number \(x\). Thus, we get that \(-1 \leq \cos(x^2 \sin(x)) \leq 1\). Dividing everything by \(e^x\), we see that

\[
-\frac{1}{e^x} \leq \frac{\cos(x^2 \sin(x))}{e^x} \leq \frac{1}{e^x}.
\]

Notice that \(\lim_{x \to +\infty} -\frac{1}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0\). So by the squeeze theorem, we get that \(\lim_{x \to +\infty} \frac{\cos(x^2 \sin(x))}{e^x} = 0\).

8. Sketch a graph of the functions below, and fill in the following limits at infinity:

\[ e^x \quad e^{-x} \quad \ln(x) \quad -\ln(x) \quad \sin(x) \quad \arctan(x) \]

For \(e^x\):

For \(e^{-x}\):

For \(\ln(x)\):

For \(-\ln(x)\):

For \(\sin(x)\):

For \(\arctan(x)\):
For \( \arctan(x) \):

(a) \( \lim_{x \to +\infty} e^x = +\infty \)
(b) \( \lim_{x \to -\infty} e^x = 0 \)
(c) \( \lim_{x \to +\infty} e^{-x} = 0 \)
(d) \( \lim_{x \to -\infty} e^{-x} = +\infty \)
(e) \( \lim_{x \to +\infty} \ln(x) = +\infty \)
(f) \( \lim_{x \to -\infty} \ln(x) = DNE \)
(g) \( \lim_{x \to +\infty} -\ln(x) = -\infty \)
(h) \( \lim_{x \to -\infty} -\ln(x) = DNE \)

(i) \( \lim_{x \to +\infty} \sin(x) = DNE \)
(j) \( \lim_{x \to -\infty} \sin(x) = DNE \)

(k) \( \lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2} \)
(l) \( \lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2} \)