(1) A function $f(x)$ is differentiable everywhere and has the following second derivative.

$$f''(x) = \frac{(2x^2 - 288)(x + 3)^9(x^2 + 25)}{20e^{16-x}}$$

Find the intervals of concavity for $f(x)$ and state each $x$-value at which the graph of $f(x)$ has an inflection point.

- $f''(x)$ is undefined nowhere
- \(e^{10-x} \neq 0\)
- $f''(x) = 0 \Rightarrow (x+3)^9 = 0 \Rightarrow x = -3$
- $(2x^2 - 288) = 0 \Rightarrow x = \pm 12$
- $x^2 + 25 \neq 0$

Find the intervals:
- $(-\infty, -3)$
- $(-3, 12)$
- $(12, \infty)$

Critical points:
- CCW: $(-12, -3), (12, \infty)$
- CDD: $(-\infty, -12), (-3, 12)$

Inflection Points:
- $x = -12, -3, 12$

(2) Determine the formula for a function $f(x)$ which satisfies the following three conditions.
- $f''(x) = 8000e^{4x} + 40 \sin(x) - 25 \cos(x)$
- $f'(0) = 80$
- $f(0) = 20$

$$f(x) = 50e^{4x} - 40 \sin(x) + 25 \cos(x) - 80x - 55$$

(3) Determine a formula for $f(x)$ given that it satisfies the following conditions.
- $f''(x) = \frac{6x^3 + x^3 - 5x + 8}{x^2 + 1}$
- $f(1) = 2\pi - 5$

$$f(x) = \frac{6}{4} x^4 - \frac{5}{2} x^2 + 8 \arctan(x) + C$$

$$f(1) = 2\pi - 5$$

$$2\pi - 5 = \frac{3}{2} (1)^4 - \frac{5}{2} (1)^2 + 8 \arctan(1) + C$$

$$2\pi - 5 = -1 + 8 (\pi/4) + C$$

$$-4 = C$$

$$f(x) = \frac{3}{2} x^4 - \frac{5}{2} x^2 + 8 \arctan(x) - 4$$
(4) Evaluate the following limit.

\[
\lim_{x \to \infty} (10e^{3x} + 8)^{5/x} \to \infty \quad \text{(Indeterminate)}
\]

\[
= \lim_{x \to \infty} e^{5 \ln (10e^{3x} + 8) / x} = \lim_{x \to \infty} e^{5 \frac{\ln (10e^{3x} + 8)}{x}} = \frac{5 \frac{1}{10e^{2x} + 8}}{70e^{3x}}
\]

\[
= \lim_{x \to \infty} e^{150e^{3x} / (10e^{3x} + 8)} = \lim_{x \to \infty} e^{450e^{3x} / 30e^{3x}} = e^1 = e
\]

(5) (4 points) Evaluate the following limit.

\[
\lim_{x \to 1} (\ln(x^6) + 1)^{1/\ln(x^3)} \to 1^\infty \quad \text{(Indeterminate)}
\]

\[
= \lim_{x \to 1} e^{\ln [(\ln(x^6) + 1)^{1/\ln(x^3)}]} = \lim_{x \to 1} e^{\frac{1}{\ln(x^3)} \cdot \ln (\ln(x^6) + 1)}
\]

\[
= \lim_{x \to 1} e^{\frac{\ln (x^6) + 1}{\ln(x^3)}} = \lim_{x \to 1} e^{\frac{1}{\ln(x^3)} \cdot \frac{x^3}{x^3 - 3x^2}} = e^1 = e
\]
(6) A function $f(x)$ is differentiable on the interval $(-\infty, \infty)$ and has the following first derivative.

$$f'(x) = 6e^{5x} (x^2 - 4)^7 (x^2 + 9)^4 (2x - 30)^{10}$$

(a) Find each critical number of the function $f(x)$.

$$X = \{-2, 2, 15\}$$

(b) State each interval upon which the function $f(x)$ is increasing.

$$(-\infty, -2), \ (2, 15), \ (15, \infty)$$

(c) State each interval upon which the function $f(x)$ is decreasing.

$$(-2, 2)$$

(d) Find the $x$-value for each local maximum value of $f(x)$.

$$X = -2$$

(e) Find the $x$-value for each local minimum value of $f(x)$.

$$X = 2$$

(7) Suppose that $g$ and $g'$ are differentiable everywhere and satisfy the following conditions.

- $g(6) = 10$
- $g'(6) = 5$
- $g''(6) = 3$

Use a linear approximation to estimate the following quantities. Simplify and write your answers in decimal form.

(a) $g(5.7)$

$$y - y_o = m (x-x_o)$$

$$y - 10 = 5 (x - 6)$$

$$y = 5x - 20$$

(b) $g'(5.7)$

$$y - y_o = m (x-x_o)$$

$$y - 5 = 3 (x - 6)$$

$$y = 3x - 13$$
(8) Use a linear approximation to obtain a good estimate for \((1006)^{2/3}\). Simplify and write your answer in decimal form.

\[
\begin{align*}
\gamma &= x^{\frac{2}{3}} = 3 \sqrt[3]{x^2} \\
\gamma &= \frac{2}{3} x^{-\frac{1}{3}} \\
\gamma'(1000) &= \frac{2}{3} \sqrt[3]{1000} = \frac{1}{15} \\
\gamma(1000) &= 3 \sqrt[3]{1000^2} = 100 \\
\gamma &= \frac{1}{15}(x-1000) \\
\gamma &= \frac{1}{15}x - 100 \\
\gamma(1006) &= \frac{1006}{15} - 100 = 100.4
\end{align*}
\]

(9) Estimate \(\sqrt{8}\) using one iteration of Newton's Method.

\[
\begin{align*}
X_2 &= X_1 - \frac{f(x_1)}{f'(x_1)} \\
X_1 &= 3 \\
X_2 &= 3 - \frac{f(3)}{f'(3)} = 3 - \frac{1}{6} = 2.5
\end{align*}
\]

(10) Fill in the missing information for the following theorems and tests.

**Mean Value Theorem** Let \(f\) be a function that satisfies the following two hypotheses.

(1) \(f\) is **continuous** on the closed interval \([a, b]\).

(2) \(f\) is **differentiable** on the open interval \((a, b)\).

Then there is a number \(c\) in \((a, b)\) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

**Rolle's Theorem** Let \(f\) be a function that satisfies the following three hypotheses.

(1) \(f\) is **continuous** on the closed interval \([a, b]\).

(2) \(f\) is **differentiable** on the open interval \((a, b)\).

(3) \(f(a) = f(b)\).

Then there is a number \(c\) in \((a, b)\) such that \(f'(c) = 0\).

**The First Derivative Test** Suppose that \(c\) is a critical number of a continuous function \(f\).

- If \(f'\) changes from positive to negative at \(c\), then \(f\) has a local \(\text{max}\) at \(c\).

- If \(f'\) changes from negative to positive at \(c\), then \(f\) has a local \(\text{min}\) at \(c\).

**The Second Derivative Test** Suppose \(f''\) is continuous near \(c\).

- If \(f'(c) = 0\) and \(f''(c) > 0\), then \(f\) has a local \(\text{min}\) at \(c\).

- If \(f'(c) = 0\) and \(f''(c) < 0\), then \(f\) has a local \(\text{max}\) at \(c\).
(11) Two vertical poles, one 4 m high and the other 16 m high, stand 15 m apart on a flat field. You must support both poles by running rope from one spot on the ground to the top of each post. What spot on the ground should you choose, in order to use the smallest amount of rope?

(12) Let \((0, 0)\) be the lower left corner and let \((x, y)\) be the upper right corner of a rectangle as shown in the diagram. The upper right corner moves along the curve \(f(x) = 25e^{-3x}\) so that its \(x\)-coordinate is moving to the right at \(4\) cm/s. How quickly is the area of the rectangle changing at the moment that the upper right corner of the rectangle has an \(x\)-coordinate of 10 cm?

(13) For each \(x > 0\), a triangle is formed with vertices \((0, 0)\), \((x, 0)\) and \((x, f(x))\) where \(f(x)\) is the function given below. What is the value of \(x\) which results in the triangle of largest area?

\[
f(x) = \frac{200}{x^2 + 4x + 25}
\]
(14) A cone-shaped coffee filter of radius 5 cm and height 10 cm contains water, which drips through a hole at the bottom at a constant rate of 1.5 cm³ per second. How quickly is the height of the water in the filter decreasing when its height is 6 cm?

(15) A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

\[ \frac{\tan \theta}{20} = \frac{y}{20} \]

\[ \text{sec}^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{20} \frac{dy}{dt} \]

\[ \frac{d\theta}{dt} = \frac{\frac{dy}{dt}}{20 \text{sec}^2 \theta} = \frac{2}{20 \left( \frac{1500}{20} \right)^2} = \frac{2}{25} \text{ rad/sec} \]

(16) There is a launch site for a hot-air balloon on the ground 20 meters away from an observer. The balloon rises vertically at a constant rate of 2 meters per second. How quickly is the angle of elevation of the balloon increasing 5 seconds after its launch?

Given \( \frac{dy}{dt} = 2 \text{ m/sec} \)

\[ \text{Want} \frac{d\theta}{dt} \text{ when } t=5 \]

\[ \tan \theta = \frac{y}{20} \]

\[ \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{20} \frac{dy}{dt} \]

\[ \frac{d\theta}{dt} = \frac{\frac{dy}{dt}}{20 \sec^2 \theta} = \frac{2}{20 \left( \frac{1500}{20} \right)^2} \]

(17) The volume of a sphere is increasing at a rate of \( 42\pi \text{ cm}^3/\text{sec} \). How quickly is the radius increasing when the diameter is 25 cm? \( V = \frac{4}{3}\pi r^3 \)

Given \( \frac{dV}{dt} = 42\pi \)

\[ \frac{d}{dt} \left( \sqrt[3]{V} \right) = \frac{d}{dt} \left( \frac{4}{3}\pi r^3 \right) \]

\[ \frac{dV}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt} \]

\[ \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{4\pi}{4\pi \left( 12.5 \right)^2} = \frac{4\pi}{4\pi \left( 12.5 \right)^2} = 0.0672 \text{ cm/sec} \]
(2) Two vertical poles, one 4 m high and the other 16 m high, stand 15 m apart on a flat field. You must support both poles by running rope from one spot on the ground to the top of each post. What spot on the ground should you choose, in order to use the smallest amount of rope? (First stop and think: what kind of problem is this? Related rates? Optimization? Differential/linear approximation?)

\[ L = \sqrt{16 + x^2} + \sqrt{256 + (15-x)^2} \quad 0 \leq x \leq 15 \]

\[ L' = \frac{2x}{2 \sqrt{16 + x^2}} + \frac{2(15-x)(-x)}{2 \sqrt{256 + (15-x)^2}} \]

\[ \frac{x}{\sqrt{16 + x^2}} = \frac{15-x}{\sqrt{256 + (15-x)^2}} \]

\[ \frac{x^2}{16 + x^2} = \frac{(15-x)^2}{256 + (15-x)^2} \]

\[ 256 + x^2(15-x)^2 = (15-x)^2 \cdot 16 + (15-x)^2 \cdot \frac{x^2}{2} \]

\[ 256x^2 = 225 - 30x + x^2 \cdot 16 \]

\[ 240 (x+5)(x-3) = 0 \]

\[ x = -5, 3 \]

\[ \text{minimum occurs at } 3 \text{ ft from shorter pole} \]

(3) A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level? (First stop and think: what kind of problem is this? Related rates? Optimization? Differential/linear approximation?)

\[ \frac{d\theta}{dt} = \frac{2\pi}{2} \text{ rad} \]

\[ \frac{d\theta}{dt} = \pi \text{ rad/min} \]

\[ \sin \theta = \frac{y}{10} \]

\[ y = 10 \sin \theta \]

\[ \frac{dy}{dt} = 10 \cos \theta \frac{d\theta}{dt} \]

\[ \frac{dy}{dt} = 10 \left( \frac{5}{13} \right) \pi = \frac{8\pi}{1} \text{ m/min} \]
Let \((0, 0)\) be the lower left corner and let \((x, y)\) be the upper right corner of a rectangle as shown in the diagram. The upper right corner moves along the curve \(y = f(x) = 25e^{-3x}\) so that its \(x\)-coordinate is moving to the right at \(4\) cm/s. How quickly is the area of the rectangle changing at the moment that the upper right corner of the rectangle has an \(x\)-coordinate of \(10\) cm?

\[
\frac{dx}{dt} = 4 \text{ cm/s}
\]

**Want:** \[
\frac{dA}{dt} \bigg|_{x=10\text{ cm}}
\]

Area = Width \times Height
\[
A = x \cdot y
\]
\[
A = x \cdot 25e^{-3x} = 25xe^{-3x}
\]
\[
\frac{d}{dt}(A) = \frac{d}{dt}(25xe^{-3x})
\]
\[
\frac{dA}{dt} = 25 \frac{dx}{dt} e^{-3x} + 25x \cdot (-3e^{-3x} \frac{dx}{dt})
\]
\[
\frac{dA}{dt} \bigg|_{x=10\text{ cm}} = 25(4)e^{-3(10)} + 25(10)(-3e^{-3(10)}) (4)
\]
\[
\frac{dA}{dt} \bigg|_{x=10\text{ cm}} = 100e^{-30} - 3000e^{-30}
\]
\[
\frac{dA}{dt} \bigg|_{x=10\text{ cm}} = -2900e^{-30} \frac{\text{cm}^2}{s}
\]
For each $x > 0$, a triangle is formed with vertices $(0,0)$, $(x,0)$ and $(x,f(x))$ where $f(x)$ is the function given below. What is the value of $x$ which results in the triangle of largest area?

$$f(x) = \frac{200}{x^2 + 4x + 25}$$

$$\text{Area} = \frac{1}{2} \text{(base)} \times \text{(height)}$$

$$A = \frac{1}{2} \cdot x \cdot f(x)$$

$$A = \frac{1}{2} \cdot x \cdot \frac{200}{x^2 + 4x + 25}$$

Maximize $A$ on $(0, \infty)$

$$A' = \frac{100(x^2 + 4x + 25) - 100x(2x+4)}{(x^2 + 4x + 25)^2}$$

$$A' = \frac{100(25-x^2)}{(x^2 + 4x + 25)^2} = \frac{100(5+x)(5-x)}{(x^2 + 4x + 25)^2}$$

$x > 0$ and $A' = 0 \Rightarrow x = 5$

Values of $A'$

$$\begin{array}{c|c|c|c|}
 x & - & 0 & + \\
 \hline
 A' & - & 0 & + \\
 \end{array}$$

$\text{abs. max, area when } x = 5$

$\Rightarrow A(5) = \frac{200}{70} = \frac{20}{7}$
A cone-shaped coffee filter of radius 5 cm and height 10 cm contains water, which drips through a hole at the bottom at a constant rate of 1.5 cm³ per second. How quickly is the height of the water in the filter decreasing when its height is 6 cm?

\[ \frac{dx}{dt} = -1.5 \text{ cm}^3/\text{s} \]

\[ \text{want: } \frac{dh}{dt} \] when \( h = 6 \text{ cm} \)

\[ V = \frac{1}{3} \pi r^2 h \]

\[ V = \frac{1}{3} \pi \left( \frac{5}{6} h \right)^2 h \]

\[ V = \frac{25}{18} h^3 \]

\[ \frac{d}{dt}(V) = \frac{d}{dt} \left( \frac{25}{18} h^3 \right) \]

\[ \frac{dV}{dt} = \frac{25}{18} h^2 \frac{dh}{dt} \]

\[ -1.5 = \frac{25}{18} (6)^2 \frac{dh}{dt} \]

\[ \frac{dh}{dt} \bigg|_{h=6 \text{ cm}} = -\frac{1.5}{\frac{25}{18} \cdot 6} = -\frac{1}{6\pi} \text{ cm/s} \]

\[ \frac{dh}{dt} \bigg|_{h=6 \text{ cm}} = -\frac{1}{6\pi} \text{ cm/s} \]

**Height is decreasing at \( \frac{1}{6\pi} \text{ cm/s} \) when height is 6 cm.**