Instructions. Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

(1) The shaded region is bounded by the curves $y = x^4 - 2x$ and $y = 3x^3 - 3x^2$ over $[0, 2]$.

(a) Give an integral (do not evaluate) for the area of the region.

We check which curve is on the top and which is on the bottom. Note at $x = 1$, we have $1^4 - 2(1) = -1$. and $3(1)^3 - 3(1)^2 = 0$. So the latter curve is on top, while the former is on bottom. The area is thus

$$\int_0^2 (3x^3 - 3x^2) - (x^4 - 2x) \, dx.$$  

(b) Give an integral (do not evaluate) for the volume obtained if this region is rotated about the line $y = -3$.

The volume is

$$\pi \int_0^2 (3x^3 - 3x^2 + 3)^2 - (x^4 - 2x + 3)^2 \, dx.$$  

(c) Give an integral (do not evaluate) for the volume obtained if this region is rotated about the line $x = 8$.

The volume is

$$2\pi \int_0^2 (8 - x) \left( (3x^3 - 3x^2)^2 - (x^4 - 2x)^2 \right) \, dx.$$  

(d) Use cylindrical shells to find the volume of a right circular cone with height $h$ and with base radius $r$. You should get $\frac{1}{3}\pi r^2 h$.

Consider the triangle in the $x - y$ plane with vertices $(0, 0)$ and $(r, 0)$ and $(0, h)$. We rotate this triangle about the $y$-axis. Note that the triangle is bounded by line passing through $(0, r)$ and $(h, 0)$ which has equation

$$f(x) = -\frac{h}{r}(x - r),$$
and the $x$ and $y$ axes. Using cylindrical shells, we have the volume is

$$2\pi \int_0^r x \left( -\frac{h}{r} (x - r) \right) \, dx = 2\pi \int_0^r -\frac{h}{r} (x^2 - rx) \, dx$$

$$= -\frac{2\pi h}{r} \left( \frac{r^3}{3} - \frac{r^3}{2} \right)$$

$$= -\frac{2\pi h}{r} \left( -\frac{r^3}{6} \right)$$

$$= \frac{\pi}{3} r^2 h.$$

The average value of a function $f(x)$ over an interval $[a, b]$ is given by the formula

$$f(x)_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x) \, dx$$

(2) Calculate the average value of $f(x) = \frac{240}{\sqrt{8x + 9}}$ on the interval $[-1, 2]$. Simplify your answer.

We have

$$f(x)_{\text{avg}} = \frac{1}{2 + 1} \int_{-1}^2 \frac{240}{\sqrt{8x + 9}} \, dx$$

$$= \frac{1}{3} \int_{-1}^2 \frac{240}{\sqrt{8x + 9}} \, dx$$

$$= 80 \int_{-1}^2 \frac{1}{\sqrt{8x + 9}} \, dx.$$

Substituting $u = 8x + 9$, with $du = 8 \, dx$, we have

$$80 \int_{-1}^2 \frac{1}{\sqrt{8x + 9}} \, dx = 80 \int_1^{25} \frac{1}{8\sqrt{u}} \, du$$

$$= 10 \int_1^{25} \frac{1}{\sqrt{u}} \, du$$

$$= 10 \int_1^{25} u^{-1/2} \, du$$

$$= 10 \left( 2(25)^{1/2} - 2(1)^{1/2} \right)$$

$$= 10(2(5) - 2(1))$$

$$= 80.$$