**Volume Integrals**

**Solutions**

**Instructions.** Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning. (Some problems can be done using the cylindrical shell method. If you already know this method DO NOT USE IT TODAY).

(1) For the following problems, set up integrals which calculate the volume of the described solid. **DO NOT EVALUATE** these integrals.

(a) The solid obtained by rotating the region bounded by \( y = x^2 - 4x + 5 \), \( x = 1 \), \( x = 4 \), and \( y = 0 \) about the \( x \)-axis.

\[
\begin{align*}
\text{Cross-section} & \quad \text{Circle} \quad A = \pi r^2 \\
radius &= y = x^2 - 4x + 5 \\
Vol &= \int_1^4 \pi (x^2 - 4x + 5)^2 \, dx
\end{align*}
\]

(b) Determine the volume of the solid obtained by rotating the region bounded by \( y = x^2 - 2x \) and \( y = x \) about the line \( y = 4 \).

\[
\begin{align*}
\text{Cross-section} & \quad \text{Washer} \quad A = \pi (r_{out}^2 - r_{in}^2) \\
r_{out} &= 4 - (x^2 - 2x) \\
r_{in} &= 4 - x \\
Vol &= \int_0^3 \pi (4 - (x^2 - 2x))^2 - \pi (4 - x)^2 \, dx
\end{align*}
\]

(c) The solid obtained by rotating the region bounded by \( y = \sqrt{x} \) and \( y = \frac{x}{4} \) that lies in the first quadrant about the \( y \)-axis.

\[
\begin{align*}
\text{Cross-section} & \quad \text{Washer} \quad A = \pi (r_{out}^2 - r_{in}^2) \\
r_{out} &= \sqrt{x} \\
r_{in} &= \frac{x}{4} \\
Vol &= \int_0^8 \pi \left((\sqrt{x})^2 - \left(\frac{x}{4}\right)^2\right) \, dy
\end{align*}
\]
(2) Let \( R \) be the region bounded by the graph of \( y = x - x^2 \) and the \( x \)-axis. Using the disk/washer method, set up an integral for the volume of the solid formed by revolving \( R \) around the \( y \)-axis. (We'll find this volume next week using the shell method. If you already know the shell method, don't use it today!)

\[
\text{Cross-section} \quad \text{Washer} \\
A = \pi (r_{\text{out}}^2 - r_{\text{in}}^2) \\
r_{\text{out}} = 1 + \sqrt{1-y} \frac{1}{2} \\
r_{\text{in}} = 1 - \sqrt{1-y} \frac{1}{2} \\
y = -x^2 + x \\
x^2 - x + y = 0 \\
x = 1 \pm \sqrt{1-y} \frac{1}{2} \\
V_{\text{Vol}} = \pi \int_{0}^{1/4} (1 + \sqrt{1-y} \frac{1}{2})^2 - (1 - \sqrt{1-y} \frac{1}{2})^2 \, dy
\]

(3) Write a definite integral that represents the following volumes.

(a) Slices perpendicular to the \( x \)-axis are squares over the area bounded by \( 2x - x^2 \) and the \( x \)-axis.

\[
\text{Cross-section} \quad \text{Square} \\
A = l^2 \\
l = y \\
x = 2x - x^2 \\
y = 2x - x^2 \\
= x(2-x) \\
V_{\text{Vol}} = \int_{0}^{2} (2x-x^2)^2 \, dx
\]

(b) Slices perpendicular to the \( x \)-axis are equilateral triangles over the area bounded by \( y = x \) and \( y = \sqrt{x} \).

\[
\text{Cross-section} \quad \text{Equilateral} \quad \Delta \\
A = \frac{1}{2} s \left( \frac{\sqrt{3}}{2} s \right) \\
S = \sqrt{x} - x \\
V_{\text{Vol}} = \int_{0}^{1} \frac{\sqrt{3}}{4} (\sqrt{x} - x)^2 \, dx
\]

(c) Slices perpendicular to the \( y \)-axis are equilateral triangles over the area bounded by \( y = x \) and \( y = \sqrt{x} \).

\[
\text{Cross-section} \quad \text{Equilateral} \quad \Delta \\
A = \frac{\sqrt{3}}{4} s^2 \\
S = y - y^2 \\
V_{\text{Vol}} = \int_{0}^{\sqrt{2}} (y-y^2)^2 \, dy
\]

(4) (Not for turning in: if there is time) Compute the volume of a "rugby ball" if it has length 300 mm and circumference 600 mm and it is formed by rotating an ellipse about its major axis (these are actual official dimensions).
\[ 2\pi b = 600 \]
\[ b = \frac{300}{\pi} \]

**Equation of Ellipsoid**

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

**Eqn of Rugby Ball, centered at origin**

\[ \frac{x^2}{(150)^2} + \frac{y^2}{(300/\pi)^2} = 1 \]

**Vertical cross-section**

**Circle**

\[ A = \pi r^2 \]

**Radius**

\[ r = \frac{300}{\pi} \sqrt{1 - \frac{x^2}{150^2}} \]

**Volume**

\[ V = \pi \int_{-150}^{150} \frac{300^2}{\pi} \left(1 - \frac{x^2}{150^2}\right) \, dx \]

\[ = \frac{300^2}{\pi} \left[ x - \frac{x^3}{3(150)^2} \right] \bigg|_{-150}^{150} \]

\[ = \frac{300^2}{\pi} \left[ \frac{150^3}{3} - \left(\frac{150^3}{3} - \left(-150^3 + \frac{150^3}{3}\right)\right) \right] \]

\[ = \frac{300^2}{\pi} \left(300 - 100\right) = \frac{300^2 \cdot 200}{\pi} \text{ mm}^3 = \frac{300^2 \cdot 200}{\pi} \text{ cm}^3 \]

\[ = \frac{60000}{\pi} \text{ cm}^3 \]