Solutions

DEVELOPMENTS AND THE SHAPE OF THE GRAPH

Instructions. Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

1. Determine the exact value of \( x \) at which \( f(x) = x^2 e^{-x} \) increases most rapidly.

\[
\text{Want Max positive slope}
\]

\[
f'(x) = x^2 \left( e^{-x} (-x) \right) + e^{-x} (2x)
= e^{-x} (2x - x^2)
\]

\[
x \mid f'(x)
\]

\[
\begin{array}{c|c}
2+\sqrt{2} & \text{+ number} \\
2-\sqrt{2} & \text{- number}
\end{array}
\]

\[
\left. \begin{array}{c}
\text{Crit Pt for Slope}
\text{f''(x)} = e^{-x} \left( 2 - 2x \right) + \left( x^2 - x \right) (-e^{-x})
= e^{-x} \left( 2 - 2x - 2x + x^2 \right)
= e^{-x} (x^2 - 4x + 2)
\end{array} \right\}
\]

\[
\text{f''(x)} = \text{undefined}
\]

\[
\text{nowhere}
\]

\[
\frac{\text{f''(x)}}{\text{= 0}}
\implies 7x^2 - 4x + 2 = 0
\implies x = \frac{2 \pm \sqrt{2}}{2}
\]

2. Find the intervals of concavity and each inflection point for the given function \( f(x) = \frac{2}{7} x^7 - 4x^6 + 15x^5 + 9x + 42. \)

\[
f'(x) = \frac{2}{7} x^7 - 4x^6 + 15x^5 + 9x + 42
\]

\[
f''(x) = 2x^6 - 24x^5 + 75x^4 + 9
\]

\[
f'''(x) = 12x^5 - 120x^4 + 300x^3
\]

\[
= 12x^3 (x^2 - 10x + 25)
\]

\[
= 12x^3 (x - 5)^2
\]

\[
\text{No inflection}
\]

\[
\text{Inflection point (0, 10)} \implies (0, 42)
\]

\[
f \text{is ccw} \quad (-\infty, 0)
\]

\[
f \text{is cww} \quad (0, 5) \text{ and } (5, \infty)
\]

\[
\text{(or (0, oo))}
\]

3. Let \( f(x) = \frac{3x}{\ln(x)} \). Determine each interval where \( f \) is increasing and each interval where \( f \) is decreasing.

\[
\text{Domain} \quad x > 0 \quad \text{and} \quad \ln(x) \neq 0
\]

\[
\text{x} \neq e^0
\]

\[
\text{at} \neq 1
\]

\[
f'(x) = \frac{\ln(x) \left( 4 - 4x \right)}{(\ln(x))^2}
\]

\[
\text{f'(x)} \text{is undefined}
\]

\[
\text{nowhere in domain}
\]

\[
\text{f(x) is up on } (e, \infty)
\]

\[
f(x) \text{ is down on } (0, 1) \text{ and } (1, e)
\]

\[
\text{Cannot combine}
\]

\[
\text{Combining}
\]

\[
f'(x) = 0
\]

\[
4 \ln(x) - 4 = 0
\]

\[
\ln(x) = 1
\]

\[
x = e
\]
4. Find the absolute maximum and minimum values for

\[ f(x) = 3x(x + 4)^{2/3} \]

on the interval \([-5, -1]\). (do not use a graphing device)

\[ f'(x) = \frac{9(x+4)^{-1/3} + 3x}{3(x+4)^{2/3} + \frac{2x}{(x+4)^{1/3}}} \]

\[ \begin{array}{c|c|c}
  x & f(x) & f'(x) \\
  \hline
  -4 & 0 & 0 \\
  -2 & 3 & \frac{2x}{(x+4)^{1/3}} \\
  0 & -5 & -2x \\
  1 & -1 & -2 \\
  \\
\end{array} \]

5. The graph of the function \(f(x)\) and its derivative \(f'(x)\) are shown. Which is bigger, \(f'(-1)\) or \(f''(-1)\)?

6. The graph below is a drawing of a derivative function \(f'(x)\). From this graph, read off the intervals where \(f(x)\) is increasing/decreasing, and list the x-coordinates where \(f(x)\) has a maximum or minimum (identify which one it is).

\[ f(x) \uparrow \text{ on } (-1, 0), \quad (2, 4) \]

\[ f(x) \downarrow \text{ on } (0, 2), \quad (4, 5) \]

7. The function below is a graph of \(f'(x)\). Use this graph to determine all inflection points of \(f(x)\), and the intervals where \(f(x)\) is concave up (ccu) or concave down (ccd).

\[ \text{Inflection Pts} \quad \lambda = 0, 2.25, 3.8, 5.5 \]

\[ \text{ccu} \quad (-1, 0), \quad (2.25, 3.8), \quad (5.5, 6) \]

\[ \text{ccd} \quad (0, 2.25), \quad (3.8, 5.5) \]

8. The rate of violent crimes in a city continues to decrease, but at a slower rate than in previous years. Letting \(f(t)\) be the rate of violent crime as a function of time, what does this tell you about \(f(t)\), \(f'(t)\), and \(f''(t)\)? Circle the correct answer below. (Hint: Try to draw a picture of a curve that fits the description above)

\[ \begin{array}{l}
  (a) \ f(t) \text{ is decreasing and concave up; } f'(t) > 0, \ f''(t) < 0 \\
  (b) \ f(t) \text{ is decreasing and concave down; } f'(t) < 0, \ f''(t) > 0 \\
  (c) \ f(t) \text{ is decreasing and concave down; } f'(t) > 0, \ f''(t) < 0 \\
  (d) \ f(t) \text{ is decreasing and concave up; } f'(t) < 0, \ f''(t) > 0 \\
\end{array} \]