**Math 221 Worksheet 12**

**Wednesday October**

**Approximations and Graphical Derivatives**

**Solutions**

Instructions. Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

1. Determine the linear approximation of $\sin(x)$ at $x = 0$. Use the graph of the function and its linear approximation to determine when this approximation is an overestimate of $\sin(x)$ and when it is an underestimate.

   \[ f(x) = \sin(x) \quad f(0) = 0 \]
   \[ f'(x) = \cos(x) \quad f'(0) = 1 \]

   \[ y = 0 = l(x - 0) \]

   \[ y = x \]

   overestimates when $x > 0$

   underestimates when $x < 0$

2. Estimate $\sqrt[3]{8.1}$.

   \[ \sqrt[3]{8} = 2 \]

   \[ \sqrt[3]{8.1} = \frac{\sqrt[3]{8.1}}{\sqrt[3]{8}} = \frac{1}{12} \]

3. A window has the shape of a square surmounted by a semi-circle. The base of the window is measured as having width 60cm with a possible error in measurement of 0.1cm. Use differentials to estimate the maximum error possible in computing the area of the window.

   \[ \text{Area} = x^2 + \frac{1}{2} \pi \left(\frac{x}{2}\right)^2 \]

   \[ A(x) = (1 + \frac{\pi}{8})x^2 \]

   \[ A'(x) = 2(1 + \frac{\pi}{8})x \]

   \[ dA = A'(x) \, dx \quad x = 60, \; dx = 0.1 \]

   \[ = 2 \left(1 + \frac{\pi}{8}\right)(60)(0.1) \]

   \[ = 12 + 3 \frac{\pi}{2} \approx 16.7 \text{ cm}^2 \]

4. Find the absolute maximum and minimum values for $f(x) = 3x(x+4)^{2/3}$ on the interval $[-5, -1]$. (do not use a graphing device)

   \[ f(x) = 3x(x+4)^{2/3} \]

   \[ f'(x) = 3(x+4)^{2/3} + 3x \cdot \frac{2}{3}(x+4)^{-1/3} \]

   \[ = 3(x+4)^{2/3} + \frac{2x}{(x+4)^{1/3}} \]

   \[ f(x) \text{ is undefined at } x = -4 \]

   \[ \frac{y}{3(x+4)^{1/3}} = -\frac{2x}{(x+4)^{1/3}} \]

   \[ 3x + 12 + 2x = 0 \]

   \[ x = -12/5 \]

   \[ \text{Abs max of 0 at } x = -4 \]

   \[ \text{Abs min of -15 at } x = -5 \]
(2) A runner sprints around a circular track of radius 100m at a constant speed of 5 m/s. The runner’s friend is standing at a distance 200m from the center of the track. How fast is the distance between the friends changing when the distance between them is 150 m? Start by drawing a picture! Hint: you will need to translate the speed of 5 m/s into a rate of change for the angle between the runner, the center of the circle, and the runner’s friend; the rate of change should be measured in radians/s.

\[
\text{Want: } \frac{dx}{dt} \text{ when } x = 150
\]

Know: Runner is moving around circle at 5 m/s

\[
L = \frac{\Theta}{2\pi} \cdot 2\pi (100) = 100 \Theta
\]

\[
\frac{dL}{dt} = 5 \text{ m/s} \quad \frac{dL}{dt} = 100 \frac{d\Theta}{dt}
\]

\[
0.5 \text{ rad/sec} = \frac{\Theta}{100} = \frac{d\Theta}{dt}
\]

\[
\sin \Theta = 0.73
\]

\[
\frac{dx}{dt} \approx 4.9 \text{ m/s}
\]

\[
(150)^2 = 50,000 = 40,000 + (200)^2 - 2(100)(200) \cos \Theta
\]

\[
2x \frac{dx}{dt} = 40,000 \left(-\sin \Theta\right) \frac{d\Theta}{dt}
\]

\[
\frac{dx}{dt} = \frac{20,000 \sin \Theta \frac{d\Theta}{dt}}{x}
\]

Plug in \( \frac{d\Theta}{dt} = 0.05 \)

\[
x = 150
\]

\[
\sin \Theta = 0.73
\]

\[
\frac{dx}{dt} \approx 4.9 \text{ m/s}
\]