New Constructions of Virtual Knot Polynomials

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Joint work with Lou Kauffman
Outline

1. Background
   - Gauss Diagrams
   - Virtual Knots
   - Linking Numbers

2. The Wriggle Number and the Wriggle Polynomial

3. Application to the Crossing Change Conjecture
   - Crossing Parity
   - Crossing Change Conjecture

4. The Affine Index Polynomial
   - Review of Quandles and Biquandles
   - Defining AIP

5. Distinguishing Mutant Knots (or not)
A knot is a smooth embedding $f : S^1 \to \mathbb{R}^3$.

We have various methods of encoding knot information in an effort to help understand them.
To every knot, we can associate a Gauss Code, which we can then draw as a Gauss Diagram. In the Gauss Code and Diagram, +/− stand for the sign of the crossing, and o/u represent whether we transverse over or under that particular crossing while traveling around the knot.

\[ \text{Gauss Code: } ao+ \text{ bu+ co+ au+ bo+ cu+} \]

![Gauss Diagram example](image)

**From Knots to Gauss Codes and Gauss Diagrams**

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From every Gauss Diagram, we can always ”try” to draw the knot diagram.
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From Gauss Codes and Diagrams to Knots
From every Gauss Diagram, we can always “try” to draw the knot diagram.

Not all Gauss Diagrams can be completed as classical knots!
Virtual Knots

We use a special symbol to denote a crossing that isn’t really there and the knot is now called a virtual knot.

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Virtual knots and links can also be described topologically as embeddings of circles in thickened surfaces (of arbitrary genus) taken up to 1-handle stabilization and and surface homeomorphisms. Here virtual crossings occur as an artifact of the planar representation. Let’s look at the Virtual Trefoil as an example.
Virtuous Trefoil
Virtual Trefoil
Two virtual diagrams are equivalent if there exists a sequence of classical and virtual Reidemeister moves connecting them. We can describe these moves with planar diagrams or with Gauss Diagrams.

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**Planar Isotopy**

**RI**

**RII**

**RIII**

**vRI**

**vRII**

**vRIII**

**Mixed**

**RIII**
A flat knot forgets all over/under information in a knot diagram. While flat classical knots are trivial, flat virtuals are highly non-trivial and can be thought of as curves on surfaces.
Classical Linking Number

Definition

Let $L$ be an ordered, classical, 2-component link. Let $C$ be the set of crossings between the 2 linked components (no self-crossings).

When traveling along the first component, $\text{Over}(C)$ (resp. $\text{Under}(C)$) is the set of crossings from $C$ we encounter as overcrossings (resp. undercrossings). The following are equivalent definitions of linking number.

1) $\text{lk}(L) = \frac{1}{2} \left[ (\# \text{ of positive crossings}) - (\# \text{ of negative crossings}) \right]$ 

2) $\text{lk}(L) = \frac{1}{2} \sum_{c \in C} \text{sign}(c)$

3) $\text{lk}(L) = \sum_{c \in \text{Over}(C)} \text{sign}(c) = \sum_{c \in \text{Under}(C)} \text{sign}(c)$
Classical vs. Virtual Linking Definitions

Overlinking Number: \((-1) + (-1)\)

Underlinking Number: \((-1) + (-1)\)
All crossings are negative

Over Linking Number: -3
Under Linking Number: -1
**Two Virtual Linking numbers**

**Definition**

For an ordered virtual 2-component link we define

Over linking number \( = \text{lk}_O(L) = \sum_{c \in \text{Over}(C)} \text{sign}(c) \)

Under linking number \( = \text{lk}_U(L) = \sum_{c \in \text{Under}(C)} \text{sign}(c) \)

Note that \( \text{Over}(C) \) and \( \text{Under}(C) \) are defined with respect to the ordering of the link.
The wriggle number for an ordered 2-component oriented link is defined as the difference between the 2 virtual linking numbers.

\[ W(L) = \sum_{c \in \text{Over}(C)} \text{sign}(c) - \sum_{c \in \text{Under}(C)} \text{sign}(c) = \text{lk}_O(L) - \text{lk}_U(L) \]

The set Over (resp. Under) is the set of crossings between the 2 linked components that we go over (resp. under) while we travel along the first component of the link diagram. We sum the signs of the crossings in these sets.
Example of Calculating Wriggle number

Let us call the link above \( L \) and calculate the wriggle number. As we travel along component 1 crossings \( a, b, c \in \text{Over}(L) \) and \( d \in \text{Under}(L) \). Therefore:

\[
W(L) = [(-1) + (-1) + (-1)] - [(-1)] = -2
\]
Obtaining a Polynomial from the Wriggle number

Definition

\[ W_K(t) = \sum_{c \in C} \text{sign}(c) t^{W(L_c)} - \text{writhe}(K) \]

We sum over all crossings, \( c \in C \) of the knot diagram, and \( L_c \) is the resulting ordered 2-component link that occurs when we do an oriented smoothing at crossing \( c \). The ordering of the links is defined by a dot convention - place a dot on the incoming overstrand to denote the first component. \( W(L_c) \) is then the wriggle number of this sublink.
Obtaining a Polynomial from the Wriggle number

Definition

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Theorem (Folwaczny-Kauffman)

The Wriggle Polynomial is an invariant of virtual knots, and is consistently zero on all classical knots.
Example of calculating the Wriggle polynomial

\[ W_K(t) = \text{sign}(a)t^{W(L_a)} + \text{sign}(b)t^{W(L_b)} - \text{writhe}(K) \]
\[ = (+1)t^{-1} + (+1)t^{+1} - 2 \]
\[ = t^{-1} + t - 2 \]
Crossing Parity

Recall that crossings in a knot diagram can be labeled as even or odd.

**Definition**

A crossing is an *even crossing* if there are an even number of terms in between the two appearances of the crossing in the Gauss Code. If there are an odd number of terms, the crossing is an *odd crossing*.
Restricting Summations to Odd Crossings

One interesting feature about odd crossings is that the *odd writhe*, \( J(K) \), is an invariant of virtual knots, where the writhe is not an invariant of classical knots (it is changed by a Redeimeister 1 move in the diagram).

\[
J(K) = \sum_{c_i \in Odd(C)} w(c_i)
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**Definition**

$\hat{W}_K(t)$, is defined as

$$\hat{W}_K(t) = \sum_{c \in Odd(C)} \text{sign}(c) t^{W(L_c)}$$
A crossing change on a non-nugatory crossing in a diagram $D$ is said to be \textit{cosmetic} if the new diagram, $D'$, is isotopic (classically or virtually) to $D$. 
Crossing Change

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Open Do non-trivial cosmetic crossing changes exist?
Crossing Change

**Definition**

A crossing change on a non-nugatory crossing in a diagram $D$ is said to be *cosmetic* if the new diagram, $D'$, is isotopic (classically or virtually) to $D$.

Open Do non-trivial cosmetic crossing changes exist? Stated as problem 1.58 of Kirby’s Problem List. There exist examples of $K_1$ and $K_2$ that differ by a single crossing change and are equivalent but their orientation is reversed. For example, $K_1 = P(3,1,-3)$ and $K_2 = P(3, -1, -3)$. 

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The following classes of knots have been shown to NOT admit cosmetic crossing changes.

<table>
<thead>
<tr>
<th>The Unknot</th>
<th>Scharlemann-Thompson (CMH, 87)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two Bridge Knots</td>
<td>I.Torisu (TAIA, 97)</td>
</tr>
<tr>
<td>Fibered Knots</td>
<td>Kalfagianni (Crelle, 11)</td>
</tr>
<tr>
<td>Genus one knots</td>
<td>Kalfagianni, Balm, Friedl, Powell (2012)</td>
</tr>
</tbody>
</table>
Results

Theorem (Folwaczny)

Switching an odd crossing in a virtual knot always results in a different knot.
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Corollary

Odd Virtuals do not admit cosmetic crossing changes.
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Odd Virtuals do not admit cosmetic crossing changes.

Proof (sketch).
Consider 2 knots, $K_+$ and $K_-$ that differ by a crossing change. Then calculate:

$$\hat{W}_{K_+}(t) - \hat{W}_{K_-}(t) = \ldots \rightarrow \ldots = t^k + t^{-k}$$
Quandles

Definition

A quandle, \( X \), is set set with a binary operation, \( * \) \((a,b) \rightarrow a*b\), such that

(I) For any \( a \in X \), \( a*a = a \)

(II) For any \( a,b \in X \), there is a unique \( c \in X \), such that \( a = c*b \)

(III) For any \( a,b,c \in X \), we have \( (a*b)*c = (a*c)*(b*c) \)
A *quandle* is an algebraic structure whose algebraic relations are equivalent to Redemeister moves in knot theory. The relation * of the quandle gives a rule for coloring the arcs in a knot diagram that is invariant under Redemeister moves (since the relation * satisfies the equivalent of those Redemeister moves in the algebraic structure).
Example (The Fox Coloring Quandles)

\((\mathbb{Z}_n, \ast)\) can be seen as a quandle with the following operation.

\[ a \ast b = 2b - a \pmod{n} \]
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The property of a knot being "colored" by a particular quandle is an invariant of a classical knot.
Examples of Quandles

Example (The Conjugation Quandles)

Any group G can be made into a quandle, $(G, *)$ with conjugation as the operation.

\[ a * b = b^{-1}ab \]
### Example (The Alexander Quandle)

If $M$ is an abelian group, and $T : M \to M$ is an automorphism, then $(M, \ast)$ becomes a quande with the following operation:

$$x \ast y = T(x) + (1 - T)(y)$$
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Example (The Alexander Quandle)

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Example (The Dihedral Quandles)

We have 2 alternate descriptions of this quandle.

1. Denoted by $R_k$, this quandle is the Alexander Quandle where the group is taken to be $\mathbb{Z}_k$ and the automorphism is multiplication by -1.
2. $R_k$ is the subquandle of $Conj(D_k)$ consisting of all reflections.
Biquandles

Using the same process we used to define a quandle, and changing our definition of arc, we can construct an invariant of virtual knots. A biquandle, $\left( S, \ast, \# \right)$ is an algebraic structure with 2 binary operations which satisfies certain axioms. A biquandle can be used to color a knot diagram as follows.

\[
\begin{align*}
& b_{\overline{a}} & a_{\overline{b}} & a^b & b^a \\
& a & b & b & a
\end{align*}
\]
Flat Biquandles

A biquandle structure on a flat knot is slightly simplified. While a biquandle must satisfy a set of 20 axioms to ensure the knot colouring is invariant under R-moves, a flat biquandle only needs to satisfy 3 axioms.

\[ b \ast a \quad a \# b \]
A new flat biquandle

Choose an arc in the knot diagram to label with 0, and continue labeling arcs in the diagram as follows.

\[(\mathbb{Z}, *, #)\] is a flat biquandle where the operations are defined as:

\[a * b = a + 1\]
\[a # b = a - 1\]
To form the polynomial, at each classical crossing we obtain a number as follows.

\[ W_+(c) = a - (b + 1) \quad W_-(c) = b - (a - 1) \]

Which number we choose to calculate above depends on whether the crossing is positive or negative. Let \( W(c) = W_{\text{sign}(c)}(c) \) The generalized Cheng Polynomial is given by the formula:

\[
P_K(t) = \sum_c \text{sign}(c) t^{W(c)} - \text{writhe}(K)
\]
Equality in Exponents

Theorem (Folwaczny-Kauffman)

*The Affine Index Polynomial equals the Wriggle Polynomial*
Equality in Exponents

**Theorem (Folwaczny-Kauffman)**

The Affine Index Polynomial equals the Wriggle Polynomial

Don’t let the choice of similar notation fool you! It is not obvious that the two different definitions of the exponents are equal.
Equality in Exponents

Theorem (Folwaczny-Kauffman)

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Don’t let the choice of similar notation fool you! It is not obvious that the two different definitions of the exponents are equal.

\[ W(L_c) = \text{the difference of virtual linking numbers after smoothing a crossing } c \text{ to obtain a link.} \]

\[ W(c) = \text{a particular difference of the labeling of the knot diagram around crossing } c \text{ by a flat biquandle.} \]
A note on Vassiliev Invariants and Mutant Knots

\[ W_{K_1}(t) = t + t^{-1} - 2 \] and \[ W_{MK_1}(t) = -t^4 + 3t - t^{-1} - 1 \]
Distinguishing a Family of Mutants

Figure: The knots $K_n$ and $MK_n$ (before and after mutation)
More on Mutation

Theorem (Folwaczny)

The Affine Index Polynomial is unable to detect mutation by positive reflection.

Proof (Sketch). Involves translating the definition of the polynomial onto the Gauss Code and analyzing set differences in arc labels.
Thank you!