NAME __________________________ Solutions

Due: At the beginning of your discussion class on Friday 10/18/19

Instructions. This quiz will be graded for accuracy, out of 10 points. All work written on this quiz must be your own. Make sure to show all your work for full credit. For every 5 minutes late the quiz is turned in, 1 point will be deducted from the final score.

1. (3 points) Find \( \lim_{x \to 1} (\ln(x^6) + 1)^{5/\ln(x^3)} \)

\[
\lim_{x \to 1} \left( \ln(x^6) + 1 \right)^{\frac{5}{\ln(x^3)}} = \lim_{x \to 1} e^{\ln \left( \left( \ln(x^6) + 1 \right)^{\frac{5}{\ln(x^3)}} \right)}
\]

\[
= e^{\lim_{x \to 1} \ln \left( \left( \ln(x^6) + 1 \right)^{\frac{5}{\ln(x^3)}} \right)}
\]

\[
= e^{\lim_{x \to 1} \frac{5 \cdot \ln \left( \ln(x^6) + 1 \right)}{\ln(x^3)}}
\]

\[
= e^{\lim_{x \to 1} \left[ \frac{5}{\ln(x^6) + 1} \cdot \frac{1}{x^6} - \frac{6x^5}{x^3} \right]}
\]

\[
= e^{\lim_{x \to 1} \frac{10}{\ln(x^6) + 1}} = e^{10}
\]
2. (4 points) Consider the following function, \( f(x) = 6xe^{-3x} + 2e^{-3x} \)

(a) Where is \( f(x) \) equal to 0 or undefined? Where is \( f(x) \) positive, negative?

\[
f(x) = 2e^{-3x}(3x + 1) \\
f(x) = 0 \\
3x + 1 = 0 \\
x = -\frac{1}{3}
\]

\[
\begin{array}{c}
\text{f(x) is positive on } (-\frac{1}{3}, \infty) \\
\text{f(x) is negative on } (-\infty, -\frac{1}{3})
\end{array}
\]

(b) What are the horizontal and vertical asymptotes of \( f(x) \)?

\[
f(x) \text{ has no VA} \\
f(x) = \frac{6x + 2}{e^{3x}}
\]

\[
\lim_{x \to \infty} \frac{6x + 2}{e^{3x}} = \frac{H}{\infty} = 0
\]

\[
\lim_{x \to -\infty} \frac{6x + 2}{e^{3x}} = -\infty
\]

\[
y = 0 \text{ is a HA}
\]

(c) Where are the critical points of \( f(x) \)? Where is \( f(x) \) increasing, decreasing?

\[
f'(x) = 6e^{-3x} + 6xe^{-3x} - 3 + 2e^{-3x}(-3)
\]

\[
= -18xe^{-3x}
\]

\[
f''(x) = \text{undefined} \\
f''(x) = 0
\]

\[
(0, 2) \text{ is a critical point}
\]

\[
\begin{array}{c}
\text{f(x) increasing } (-\infty, 0) \\
\text{f(x) decreasing } (0, \infty)
\end{array}
\]

(d) Where is \( f''(x) = 0 \)? Where is \( f(x) \) concave up, concave down?

What, if any, are the inflection points of \( f(x) \)?

\[
f''(x) = -18e^{-3x} + 18 \frac{e^{-3x}(-3)}{-1 + 3x}
\]

\[
= 18e^{-3x}(-1 + 3x)
\]

\[
f''(x) = \text{undefined} \\
f''(x) = 0
\]

\[
-1 + 3x = 0 \\
x = \frac{1}{3}
\]

\[
\begin{array}{c}
\text{f(x) is concave up } (\frac{1}{3}, \infty) \\
\text{f(x) is concave down } (-\infty, \frac{1}{3})
\end{array}
\]

\[
(\frac{1}{3}, \frac{4}{e}) \text{ is an inflection point}
\]

(e) Use the information above to sketch a quick graph of \( f(x) \).
3. (3 points) Suppose \( f(x) = \frac{1}{2} \ln(x) \) and \( g(x) = -\frac{1}{8} \ln(x) \).

For each \( x \) in the interval \((0, 1)\), consider the rectangle formed in the following manner.

- Its right side is the line segment connecting the points \((x, f(x))\) and \((x, g(x))\).
- Its left side lies along the \(y\)-axis.

Which value of \( x \) in the interval \((0, 1)\) results in the rectangle of largest area?

\[
\begin{align*}
\text{Area} &= (\text{width})(\text{height}) \\
A &= x \cdot (g(x) - f(x)) \\
A &= x \left( -\frac{1}{8} \ln(x) - \frac{1}{2} \ln(x) \right) \\
A &= -\frac{7}{10} x \cdot \ln(x)
\end{align*}
\]

Want: Maximize \( A = -\frac{7}{10} x \ln(x) \) on \((0, 1)\)

\[
\begin{align*}
A' &= -\frac{7}{10} \ln(x) + \left( -\frac{7}{10} x \cdot \frac{1}{x} \right) \\
A' &= -\frac{7}{10} (\ln(x) + 1) \\
\frac{A}{1} &= 0 \quad \ln(x) + 1 = 0 \\
\ln(x) &= -1 \\
x &= e^{-1} \\
x &= \frac{1}{e}
\end{align*}
\]

Without a calculator, use test points of \(\frac{1}{e}\) and \(\frac{1}{e^2}\)

\[
\begin{align*}
A' \left( \frac{1}{e} \right) &= - \quad A' \left( \frac{1}{e^2} \right) = + \\
\frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}}
\end{align*}
\]

\(x = \frac{1}{e}\) is a local max and the only critical point on an open interval, so it is also an absolute max on the open interval.

The maximum area occurs when \(x = \frac{1}{e}\)