Mock Midterm 2B

Note: The problems on this mock midterm have not necessarily been selected to allow them to be easy to work without a calculator. The problems on the real midterm will not require the use of a calculator.

(1) Given \( g(t) = \frac{5t^2}{t^2 - 3t - 4} \).
   
   (a) What are the asymptotes for \( g(t) \)?
   
   (b) Find and classify all critical points of \( g(t) \).

   (c) Find the inflection points and intervals of concavity for \( g(t) \).

   (d) Sketch a graph of \( g(t) \). Be sure to include all important features of the graph.

(a) To find asymptotes for \( g(t) \), we look for the points where the denominator is 0. We have \( t^2 - 3t - 4 = (t - 4)(t + 1) = 0 \) and so the denominator is 0 when \( t = -1 \) or \( t = 4 \). We need to check the behavior of the function as \( t \) approaches these values. We have

\[
\begin{align*}
\lim_{t \to -1^-} \frac{5t^2}{(t - 4)(t + 1)} &= \frac{5}{0^-} = +\infty \\
\lim_{t \to -1^+} \frac{5t^2}{(t - 4)(t + 1)} &= \frac{5}{0^-} = -\infty \\
\lim_{t \to 4^-} \frac{5t^2}{(t - 4)(t + 1)} &= \frac{80}{0^-} = -\infty \\
\lim_{t \to 4^+} \frac{5t^2}{(t - 4)(t + 1)} &= \frac{80}{0^+} = +\infty 
\end{align*}
\]

So we have vertical asymptotes at \( t = -1 \) and \( t = 4 \). We check for horizontal asymptotes by checking the limiting behavior of the function as \( t \) approaches \( \pm \infty \). We have

\[
\begin{align*}
\lim_{t \to -\infty} \frac{5t^2}{t^2 - 3t - 4} &= \lim_{t \to -\infty} \frac{5t^2}{t^2 - 3t - 4} \frac{1}{t^2} \\
&= \lim_{t \to -\infty} \frac{5}{1 - \frac{3}{t} - \frac{4}{t^2}} \\
&= 5 \\
\lim_{t \to +\infty} \frac{5t^2}{t^2 - 3t - 4} &= \lim_{t \to +\infty} \frac{5t^2}{t^2 - 3t - 4} \frac{1}{t^2} \\
&= \lim_{t \to +\infty} \frac{5}{1 - \frac{3}{t} - \frac{4}{t^2}} \\
&= 5 
\end{align*}
\]

and so \( g(t) \) has a horizontal asymptote at \( y = 5 \).
(b) To find the critical points of \(g(t)\), we need to differentiate and set to zero. We should also recall that points of discontinuity in \(g(t)\) might lead to changes in sign of our derivative. We have

\[
g'(t) = \frac{10t \cdot (t^2 - 3t - 4) - 5t^2 \cdot (2t - 3)}{(t^2 - 3t - 4)^2} = \frac{-15t^2 - 40t}{(t^2 - 3t - 4)^2} = \frac{-5t(3t + 8)}{(t^2 - 3t - 4)^2}
\]

Observe that this can be 0 only when the numerator is 0, and hence we have the critical numbers \(t = 0\) and \(t = -8/3\). Note also that the derivative is undefined at \(t = -1\) and \(t = 4\), and hence these are also places that we need to check for a change in sign in the derivative. We obtain the following number line:

\[
\begin{array}{cccc}
\text{x = -8/3} & \text{x = -1} & \text{x = 0} & \text{x = 4} \\
\end{array}
\]

It follows from this that we have a local minimum at \((-8/3, g(-8/3)) = (-8/3, 16)\) and a local maximum at \((0, g(0)) = (0, 0)\).

(c) To find the intervals of concavity, we need to find the second derivative and set it equal to 0. We have

\[
g''(t) = \frac{(-30t - 40)(t^2 - 3t - 4)^2 - (-15t^2 - 40t)(2)(t^2 - 3t - 4)(2t - 3)}{(t^2 - 3t - 4)^4} = \frac{(t^2 - 3t - 4)[(-30t - 40)(t^2 - 3t - 4) - (-15t^2 - 40t)(2)(2t - 3)]}{(t^2 - 3t - 4)^4} = \frac{30t^3 + 120t^2 + 160}{(t^2 - 3t - 4)^3}
\]

We can use software to see that this has a root at \(t \approx -4.83\) (don’t worry if you don’t see it, factoring something like this would not be fair game for the exam). Our number line for concavity then looks like the following line:

\[
\begin{array}{cccc}
\text{x = -4.83} & \text{x = -1} & \text{x = 0} & \text{x = 4} \\
\end{array}
\]

and we see that we have an inflection point at \((-4.83, g(-4.83))\).
(2) An environmental study for a certain community indicates that there will be $Q(p) = p^2 + 4p + 900$ units of a harmful pollutant in the air when the population is $p$ thousand people. If the population is currently 50,000 and is increasing at the rate of 1500 per year, at what rate is the level of pollution increasing?

Observe that this is a related rates problem. We are interested in how $Q$ is changing over time when $p$ is also changing with time. Hence, we need to differentiate here with respect to time, rather than population. Also observe that since our equation uses population in thousands, we have $p = 50$ and $p' = 1.5$. Then

$$\frac{d}{dt}[Q(p)] = \frac{d}{dt}[p^2 + 4p + 900]$$

$$\frac{dQ}{dt} = 2p \frac{dp}{dt} + 4 \frac{dp}{dt}$$

$$\bigg|_{p=50, p'=1.5} = 2(50)(1.5) + 4(1.5)$$

$$= 150 + 6 = 156$$

and hence the pollution is rising at the rate of 156 units of pollutant per year.

(3) When a particular commodity is priced at $p$ dollars per unit, consumers demand $q = \sqrt{400 - 0.01p^2}$.

(a) Find the elasticity of demand for this commodity.

(b) For a unit price of $120$, is the demand elastic, inelastic, or of unit elasticity?
(a) 
\[
E(p) = -\frac{p dq}{q dp} = -\frac{p}{\sqrt{400 - 0.01p^2}} \cdot \frac{-0.01p}{\sqrt{400 - 0.01p^2}} = \frac{0.01p^2}{400 - 0.01p^2}
\]

(b) If \( p = 120 \), our elasticity of demand is given by 
\[
E(120) = \frac{0.01 \cdot 120^2}{400 - 0.01 \cdot 120^2} = \frac{144}{256} < 1
\]
and so our demand is inelastic at \( p = 120 \).

(4) Suppose the total cost in dollars of manufacturing \( q \) units is \( C(q) = 3q^2 + q + 500 \). 
(a) Use marginal analysis to estimate the cost of manufacturing the 41st unit. 
(b) Compute the actual cost of manufacturing the 41st unit.

(a) To use marginal analysis, we really just want to figure out what the value of the derivative is at \( q = 40 \). We have 
\[
C'(q) = 6q + 1 \Rightarrow C'(40) = 241
\]
so we estimate the cost of making the 41st unit at $241.
(b) To compute the actual cost of manufacturing the 41st unit, we need to compute the cost of manufacturing 40 units and subtract it from the cost of manufacturing 41 units. We have 
\[
C(41) - C(40) = 5584 - 5340 = 244
\]

(5) Carla is a carpenter who has been hired to make a closed box with a square base and volume of 250 cubic meters. The material for the top and bottom of the box costs $2 per square meter, and the material for the sides costs $1 per square meter. Can Carla construct the box for less than $300? What is the price of the least expensive box?

![Box Diagram]
We begin by drawing a picture of our box as at above, and labeling the important parts. We are really looking to minimize the price of a box of fixed volume. We hope that this minimum price will be less than $300. This means that we have

\[ V = 250 = x^2h \Rightarrow h = \frac{250}{x^2} \]

We wish to minimize cost, which is given by

\[ C = 2(2x^2) + 4(xh) \Rightarrow C = 4x^2 + \frac{1000}{x} \]

We wish to minimize this. We have

\[ C'(x) = 8x - 1000/x^2 = 0 \Rightarrow 8x = 1000/x^2 \Rightarrow 8x^3 = 1000 \Rightarrow x = 5 \]

Now, let’s see what the total cost is when \( x = 5 \). We have

\[ C(5) = 4(25) + \frac{1000}{5} = 100 + 200 = 300 \]

So the least expensive box that Carla can construct will cost exactly $300. She cannot construct a less expensive box that meets the requirements.