Mock Midterm 1A

Note: The problems on this mock midterm were not necessarily selected to allow them to be easy to work without a calculator. The problems on the real midterm will not require a calculator.

(1) (a) Give the definition of the derivative.

The derivative of the function \( f(x) \) with respect to \( x \) is the function \( f'(x) \) given by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

(b) Use the definition to compute the derivatives of

(i) \( f(x) = \frac{1}{x^2} \) (Hint: You’ll need to find a common denominator for the top.)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{1}{(x+h)^2} \frac{x^2 - 1}{x^2 (x+h)^2}
= \lim_{h \to 0} \frac{x^2 - x^2 - 2xh - h^2}{hx^2(x + h)^2}
= \lim_{h \to 0} \frac{-2xh - h^2}{hx^2(x + h)^2}
= \lim_{h \to 0} \frac{-2x - h}{x^2(x + h)^2}
= \frac{-2x - 0}{x^2(x + 0)^2}
= \frac{-2x}{x^4}
= \frac{-2}{x^3}
\]

Observe that this agrees with the derivative computed using the power rule.
(ii) \( g(x) = \sqrt{x} \) (Hint: Multiply by the conjugate \( \sqrt{x + h} + \sqrt{x} \).)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}
= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}
= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}
= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}
= \lim_{h \to 0} \frac{x}{(\sqrt{x + h} + \sqrt{x})}
= \frac{1}{\sqrt{x + 0} + \sqrt{x}}
= \frac{1}{2\sqrt{x}}
\]

Note: Hints will not be provided on the actual midterm.

(2) Compute the derivatives of the following functions using any method. You need not simplify the results.
(a) \( f(x) = \frac{1}{\sqrt{x}} \)

\[
f'(x) = (x^{-1/3})'
= (-1/3)x^{-1/3 - 1}
= (-1/3)x^{-4/3}
\]

(b) \( g(x) = (x^2 + 3)(1/x - x^3) \)

\[
g'(x) = (x^2 + 3)'(x^1 - x^3) + (x^2 + 3)(x^{-1} - x^3)'
= (2x)(x^1 - x^3) + (x^2 + 3)(-x^{-2} - 3x^2)
\]
You may stop here.
\[
= 2 - 2x^4 + [-1 - 3x^4 - 3x^{-2} - 9x^2]
= -3x^{-2} + 1 - 9x^2 - 5x^4
\]
(c) \( h(x) = \frac{(x^2 + 3)}{(\sqrt{x} - x^3)} \)

\[
    h'(x) = \frac{(x^2 + 3)(x^{1/2} - x^3) - (x^2 + 3)(x^{1/2} - x^3)'}{(x^{1/2} - x^3)^2} = \frac{2x(x^{1/2} - x^3) - (x^2 + 3)([1/2]x^{-1/2} - 3x^2)}{(x^{1/2} - x^3)^2}
\]

(3) During the summer, a group of students builds kayaks in a converted garage. The rental for the garage is $1500 for the summer, and the materials needed to build a kayak cost $125. The kayaks can be sold for $275 apiece.

We begin by finding a function relating the group’s profit to the number of kayaks they sell. Let \( k \) be the number of kayaks sold. Then we have

\[
    R(k) = \text{price per kayak}(kayaks sold) = 275k,
    
    C(k) = \text{fixed cost} + \text{(cost per kayak)(kayaks made)} = 1500 + 125k,
    
    P(k) = R(k) - C(k) = 275k - (1500 + 125k) = 150k - 1500.
\]

(a) How many kayaks must the students sell to break even?

\[
    0 = P(k) = 150k - 1500 \Rightarrow 150k = 1500 \Rightarrow k = 10
\]

(b) How many kayaks must the students sell to make a profit of at least $1000?

\[
    1000 \leq P(k) = 150k - 1500 \Rightarrow 2500 \leq 150k \Rightarrow 50/3 \leq k
\]

and so they must sell 17 kayaks to make a profit of at least $1000.

(4) Specify the domain of each of these functions.

(a) \( f(x) = x^2 - 2x + 6 \)

Since \( f(x) \) is a polynomial, the domain is all real numbers.

(b) \( g(x) = \frac{(x - 3)}{(x^2 + x - 2)} \)

The domain of \( g(x) \) is all real numbers except for those points where the denominator is zero. Since we can factor

\[
    x^2 + x - 2 = (x + 2)(x - 1) \Rightarrow x = -2, x = 1
\]

we have that the domain of \( g(x) \) is all real numbers except \( x = -2 \) and \( x = 1 \).

(c) \( h(x) = \sqrt{x^2 - 9} \)

The square root function is defined only for nonnegative values, so we need

\[
    x^2 - 9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow x \geq 3 \text{ or } x \leq -3.
\]

(5) Compute the following limits. If the limit is infinite, indicate whether it is \(+\infty\) or \(-\infty\).

(a) \( \lim_{x \to 3^+} \sqrt{3x - 9} \)

Since our function is defined for \( x \geq 3 \), we can plug in \( x = 3 \) to find that our limit is 0
(b) \( \lim_{x \to 5^+} \frac{\sqrt{2x - 1} - 3}{x - 5} \)

\[
\lim_{x \to 5^+} \frac{\sqrt{2x - 1} - 3}{x - 5} = \lim_{x \to 5^+} \frac{\sqrt{2x - 1} - 3}{x - 5} \cdot \frac{\sqrt{2x - 1} + 3}{\sqrt{2x - 1} + 3} \\
= \lim_{x \to 5^+} \frac{(\sqrt{2x - 1})^2 - 3^2}{(x - 5)(\sqrt{2x - 1} + 3)} \\
= \lim_{x \to 5^+} \frac{2x - 1 - 9}{(x - 5)(\sqrt{2x - 1} + 3)} \\
= \lim_{x \to 5^+} \frac{2(x - 5)}{(x - 5)(\sqrt{2x - 1} + 3)} \\
= \lim_{x \to 5^+} \frac{2}{\sqrt{2 \cdot 5 - 1} + 3} \\
= \frac{2}{3 + 3} = \frac{1}{3}
\]

(c) \( \lim_{x \to 0^+} x - \sqrt{x} \)

Since both \( x \) and \( \sqrt{x} \) are defined for \( x \geq 0 \), we can simply plug in 0 for \( x \) to obtain

\[
\lim_{x \to 0^+} x - \sqrt{x} = 0 - \sqrt{0} = 0.
\]

(6) A city recreation department plans to build a rectangular playground 3600 square meters in area. The playground is to be surrounded by a fence. Express the length of the fencing as a function of the length of one of the sides of the playground, draw the graph, and estimate the dimensions of the playground requiring the least amount of fencing.

Let \( x \) be the width of the playground and \( y \) be the length of the playground. Then the perimeter of the playground (i.e. the amount of fencing required) is given by \( P = 2x + 2y \) and the area of the playground is \( A = xy = 3600 \). Solving the equation for area for \( y \), we have

\[
y = \frac{3600}{x}.
\]

Substituting back into our function for perimeter, we have

\[
P(x) = 2x + 2\left(\frac{3600}{x}\right) = 2x + \frac{7200}{x}.
\]

A graph of this function looks like:
From the graph, it appears that the minimum fencing cost will occur when $x \approx 60$, in which case $y = \frac{3600}{60} = 60$. 