Math 234 Spring 2014 Exam 3 Version 1  
Monday, April 21, 2014

Name: ___________________________ UIN: ___________________________

Circle the section you are registered for:

ADA (9:00 – Neriman)  BDA (9:00 – Dileep)  CDA (8:00 – Nima)
ADB (10:00 – Neriman) BDB (10:00 – Dileep)  CDB (9:00 – Nima)
ADC (11:00 – Itziar)  BDC (11:00 – Jiashun) CDC (11:00 – Michael)
ADD (12:00 – Itziar)  BDD (12:00 – Sean)  CDD (12:00 – Michael)
ADE (1:00 – Malik)   BDE (1:00 – Argen)   CDE (1:00 – Jaehoon)
ADF (2:00 – Malik)   BDF (2:00 – Argen)   CDF (2:00 – Claire)
ADG (3:00 – Alessandro) BDG (3:00 – Albert) CDG (3:00 – Claire)
ADH (4:00 – Alessandro) BDH (4:00 – Albert) CDH (11:00 – Sean)

1) You should have already checked into the exam and had your calculator approved.  
   Failure to have checked in will result in your receiving a score of zero on this exam.
2) No baseball caps, hoodies, etc. or dark sunglasses. All hats are to be removed.
3) All book bags should be closed and placed at the front of the classroom. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out NOW.
4) No cells phones. Turn them off now. If you are seen with a cell phone in hand during the exam it will be considered cheating and you will be asked to leave. This includes using it as a time-piece.
5) No iPods or MP3s players, etc. Same rules as with cell phones.
6) If you have a question, raise your hand and a proctor will come to you. If you have to use the bathroom, do so NOW. You will not be permitted to leave the room and return during the exam.
7) Every exam is worth a total of 100 points. Check to see that you have all of the pages. Including this cover sheet, each exam has 9 sides. Each problem is worth 12 points. You will receive 4 point for your name and section.
8) Be sure to print your proper name clearly at the top of this page. Also circle the section for which you are registered.
9) If you finish early, quietly and respectfully get up and hand in your exam.
10) When time is up, you will be instructed to put down your writing utensil, close your exam and remain seated. Anyone seen continuing to write after time is called will have their exam marked and lose all points on the page they are writing on.
11) To ensure that you receive full credit, show all of your work.
12) Good luck. You have 60 minutes to complete this exam.

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(1) Suppose that a demand function is given by
\[ q(p) = 750 - 10p^2 \]

(a) Find the elasticity of demand.
(b) Determine what price \( p \) should be charged to maximize revenue.
(c) Where does our demand function have unit elasticity?
(d) For what values of \( p \) is our demand elastic? Inelastic?

(a) The elasticity of demand is given by
\[
E = \frac{-p}{q(p)} \cdot \frac{dq}{dp}
\]
\[
= \frac{-p}{750 - 10p^2} \cdot -20p
\]
\[
= \frac{20p^2}{750 - 10p^2}
\]

(b) We know that revenue is maximized when the elasticity of demand is 1. We have
\[
\frac{20p^2}{750 - 10p^2} = 1
\]
\[
20p^2 = 750 - 10p^2
\]
\[
30p^2 = 750
\]
\[
p^2 = 25
\]
\[
p = 5
\]

So our revenue is maximized when the price charged is $5.

(c) We’ve already solved this one in part (b). The revenue is maximized at the point of unit elasticity, and so our revenue is maximized when the price is $5.

(d) Recall that our demand is elastic when \( E(p) > 1 \) and inelastic when \( E(p) < 1 \). We need to figure out where this is true. It will suffice to determine whether the demand is elastic or inelastic for one price on each side of $5. When \( p = 4 \), we have
\[
E(4) = \frac{20(4)^2}{750 - 10(4)^2} = \frac{320}{590} < 1
\]
So the demand is inelastic for \( p < 5 \). Similarly, when \( p = 6 \), we have
\[
E(6) = \frac{20(6)^2}{750 - 10(6)^2} = \frac{720}{390} > 1
\]
So the demand is elastic for \( p > 5 \).
(2) Given the relation \( x^2 - 4y^2 = 3x^3y^4 \)

(a) Find \( \frac{dy}{dx} \).

(b) The graph of the relation goes through the points \((-1, 1)\) and \((-1, -1)\). Is it a function?

(c) Find the equation of the tangent line to the graph at the point \((-1, 1)\).

(a) We use implicit differentiation to find \( \frac{dy}{dx} \). We have

\[
2x - 8y \cdot \frac{dy}{dx} = 9x^2y^4 + 12x^3y^3 \cdot \frac{dy}{dx}
\]

\[
2x - 9x^2y^4 = (12x^3y^3 + 8y)\frac{dy}{dx}
\]

\[
\frac{2x - 9x^2y^4}{12x^3y^3 + 8y} = \frac{dy}{dx}
\]

(b) The relation is not a function because it will fail the vertical line test. The same value of \(x\) yields two different values of \(y\).

(c) To find the equation of the tangent line at \((-1, 1)\), we first need to know the slope of the tangent line at that point. To determine this, we will evaluate \( \frac{dy}{dx} \) at the point \((-1, 1)\). We have

\[
\left. \frac{dy}{dx} \right|_{(-1,1)} = \frac{2(-1) - 9(-1)^2(1)^4}{12(-1)^3(1)^3 + 8(1)} = \frac{-2 - 9}{-12 + 8} = \frac{11}{4}
\]

Using point slope form, we find that

\[
y - 1 = \frac{11}{3}(x - 1) \implies y = \frac{11}{4}x + \frac{15}{4}
\]
(3) Find the absolute extrema of \( f(x) = 2x^3 - 3x^2 + 3 \) on the interval \([-1, 2]\).

We begin by finding the derivative and checking for critical points. We have
\[
    f'(x) = 6x^2 - 6x = 6x(x - 1)
\]
which has critical points at \( x = 0 \) and \( x = 1 \). To determine which points yield the absolute extrema, we must evaluate the function at both of our critical points, as well as at the endpoints of our interval. We have

<table>
<thead>
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<th>( x )</th>
<th>( f(x) )</th>
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<tr>
<td>-1</td>
<td>( 2(-1)^3 - 3(-1)^2 + 3 = -2 - 3 + 3 = -2 )</td>
</tr>
<tr>
<td>0</td>
<td>( 2(0)^3 - 3(0)^2 + 3 = 0 - 0 + 3 = 3 )</td>
</tr>
<tr>
<td>1</td>
<td>( 2(1)^3 - 3(1)^2 + 3 = 2 - 3 + 3 = 2 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2(2)^3 - 3(2)^2 + 3 = 16 - 12 + 3 = 7 )</td>
</tr>
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So our absolute maximum of \( y = 7 \) occurs when \( x = 2 \) and our absolute minimum of \( y = -2 \) occurs when \( x = -1 \).
(4) Evaluate the following:

(a) \( \int 2xe^{x^2+1} \, dx \)

We begin by using the substitution \( u = x^2 + 1 \). Then \( du = 2x \, dx \), and so our problem becomes

\[
\int 2xe^{x^2+1} \, dx = \int e^u \, du = e^u + C = e^{x^2+1} + C
\]

(b) \( \int \frac{7x^6 - 6}{x^7 - 6x + 3} \, dx \)

We begin by using the substitution \( u = x^7 - 6x + 3 \). Then \( du = 7x^6 - 6 \, dx \), and so our problem becomes

\[
\int \frac{7x^6 - 6}{x^7 - 6x + 3} \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|x^7 - 6x + 3| + C
\]
A packing company is designing a box with a square base and no top. The volume is to be 32 $m^3$. To reduce cost, the box should have minimum surface area. What dimensions (height, length, width) should the box have?

We have the following diagram:

![Diagram of a box with a square base and no top](image)

Observe that our volume is given by

$$V = x^2h = 32 \implies h = \frac{32}{x^2}$$

Furthermore, the surface area is given by

$$SA = 4xh + x^2 \implies SA = 4x\left(\frac{32}{x^2}\right) + x^2 = 128x^{-1} + x^2$$

We need to find the critical points of $SA$ and determine which of them is our absolute maximum. We have

$$SA'(x) = -128x^{-2} + 2x = 0$$

$$2x = \frac{128}{x^2}$$

$$2x^3 = 128$$

$$x^3 = 64$$

$$x = 4$$

So our surface area will be minimized when the base is 4 m by 4 m. In order to have a total volume of 32 $m^3$ we will need to have a height of

$$h = \frac{32}{4^2} = 2$$

and so the box with minimum surface area will have dimensions $4 \times 4 \times 2$.  

(6) A boat leaves a given point and travels north at 8 mph. Another boat leaves the same point at the same time and travels east at 6 mph. At what rate is the distance between the 2 boats changing at the instance the boats have traveled for 1 hour?

We have the following diagram:

And we are given that

\[
\frac{dx}{dt} = 8 \quad \text{and} \quad \frac{dy}{dt} = 6
\]

Differentiating \( D(x, y) \) with respect to \( t \), we find

\[
D(x) = (x^2 + y^2)^{1/2}
\]

\[
\frac{dD}{dt} = \frac{1}{2} (x^2 + y^2)^{-1/2} \cdot \left( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right)
\]

After 1 hour, we have that \( x(1) = 8 \) and \( y(1) = 6 \). Plugging these, along with our known derivatives into our expression we have

\[
\frac{dD}{dt} = \frac{1}{2} \left( 8^2 + 6^2 \right)^{-1/2} \cdot (2 \cdot 8 \cdot 8 + 2 \cdot 6 \cdot 6)
\]

\[
= \frac{1}{2} \cdot 100^{-1/2} \cdot (200)
\]

\[
= 10
\]
(7) Compute the following indefinite integrals.

(a) \[ \int 9 \, dy \]

\[ \int 9 \, dy = 9y + C \]

(b) \[ \int 5x^2 - 6x + \frac{1}{x} \, dx \]

\[ \int 5x^2 - 6x + \frac{1}{x} \, dx = \frac{5}{3}x^3 - 3x^2 + \ln|x| + C \]

(c) \[ \int \frac{\sqrt{x} + 1}{\sqrt{x}} \, dx \]

\[ \int \frac{\sqrt{x} + 1}{\sqrt{x}} \, dx = \int \frac{\sqrt{x} + 1}{\sqrt{x}} \, dx \]
\[ = \int 1 + x^{-1/2} \, dx \]
\[ = x + 2x^{1/2} + C \]

Note: If you do this using substitution, using \( u = \sqrt{x} + 1 \), then you will end up with

\[ \int \frac{\sqrt{x} + 1}{\sqrt{x}} \, dx = (1 + \sqrt{x})^2 + C \]

(d) \[ \int \frac{\pi^3}{y^3} + e^x \, dy \]

\[ \int \frac{\pi^3}{y^3} + e^x \, dy = \int \pi^3 y^{-3} + e^x \, dy \]
\[ = -\frac{\pi^3}{2} y^{-2} + e^x y + C \]
\[ = -\frac{\pi^3}{2y^2} + e^x y + C \]
Musk the friendly pit bull is visited by her long-time friend, Daphne the grumpy poodle. Musk and Daphne take off running into the sunset together. Their velocity for the first 4 seconds of their romp is given by the graph below. (You may assume that the numbers along the bottom and left edges give accurate information about points on the graph.) Using four subintervals, you will give 2 estimates for the total distance they traveled during the 4 seconds (from t = 0 to t = 4), once with a left endpoint approximation, and once with a right endpoint approximation, by doing the following:

(a) Use the graphs provided to draw the approximation rectangles from the left and right sides.
(b) Compute the given approximations.
(c) Which approximation gives you an underestimate? an overestimate?
Our computations then become

\[
\text{LEFT} = 1 \cdot 0 + 1 \cdot 5 + 1 \cdot 8 + 1 \cdot 10 = 23
\]

and

\[
\text{RIGHT} = 1 \cdot 5 + 1 \cdot 8 + 1 \cdot 10 + 1 \cdot 12 = 35
\]

It is clear from the pictures that the left endpoint approximation gives you an underestimate and the right endpoint approximation gives you an overestimate.