(1) (10 points) Given the relation \(2xy^2 - y = 10\)

(a) (5 points) Find \(\frac{dy}{dx}\)

\[2x \cdot 2y \frac{dy}{dx} + 2y^2 - \frac{dy}{dx} = 0.\]

\[\frac{dy}{dx} = \frac{-2y^2}{4xy - 1}\]

(b) (5 points) Find the equation of the tangent line to the graph at the point \(x=0\).

\[x = 0 \quad y = -10 \quad +2\]

\[\frac{dy}{dx} = \frac{-200}{1} = 200 \quad +2\]

\[y = 200x - 10 \quad +1\]

2
(1) (10 points) Given the relation \(2xy^2 - y = 10\)

(a) (5 points) Find \(\frac{dy}{dx}\)

1 pt for each of 3

2 pts for finish

-1 pt if they write \(\frac{dy}{dx} = \ldots\)

but got wrong answer

(b) (5 points) Find the equation of the tangent line to the graph at the point \(x=0\).

Finding \(y\) \[+1\] plug in 0 but make an error admirable

Finding \(\frac{dy}{dx}\) \[+1\] plug in \(y\) they got and \(\frac{dy}{dx}\) they got

Finding \(\frac{dy}{dx}\) \[ +2 \] actual answer +2

Use point slope form w/ whatever they got -1
(1) (10 points) Given the relation \(5xy^2 - y^3 = 5\)

(a) (5 points) Find \(\frac{dy}{dx}\)

\[
5x \cdot 2y \frac{dy}{dx} + 5y^2 - \frac{dy}{dx} = 0 + 3
\]

\[
(10xy - 1) \frac{dy}{dx} = -5y^2
\]

\[
\frac{dy}{dx} = \frac{-5y^2}{10xy - 1}
\]

(b) (5 points) Find the equation of the tangent line to the graph at the point \(x = 0\).

\[
x = 0 + 2
\]

\[
y = -5
\]

\[
\frac{dy}{dx} = \frac{-5(-5)^2}{-1} = 125 + 2
\]

\[
y = 125x + 5 + 1
\]
(1) (10 points) Given the relation $5xy^2 - y^2 = 5$

(a) (5 points) Find $\frac{dy}{dx}$

\[
5x \cdot 2y \frac{dy}{dx} + 5y^2 - 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{5y^2}{10xy - 2y} = -\frac{5y}{10x - 2}
\]

(b) (5 points) Find the equation of the tangent line to the graph at the point $x = 0$.

\[
x = 0, \quad y = \sqrt{5}
\]

\[
m = \frac{-\frac{8\sqrt{5}}{2}}{-2} = \frac{4\sqrt{5}}{2} = \frac{5\sqrt{5}}{2}
\]

\[
y + \sqrt{5} = -\frac{5\sqrt{5}}{2} x + 1
\]

\[
y = \frac{5\sqrt{5}}{2} x + \sqrt{5}
\]

+1 if get one right line.
(1) (10 points) Given the relation \(5xy^2 - y^2 = 5\)

(a) (5 points) Find \(\frac{dy}{dx}\)

\[
\frac{d}{dx} (5xy^2 - y^2) = \frac{d}{dx} (5) \\
(5x) \left(2y \frac{dy}{dx}\right) + y^2 (5) - 2y \frac{dy}{dx} = 0 \\
10xy \frac{dy}{dx} - 2y \frac{dy}{dx} = -5y^2 \\
\frac{dy}{dx} (10xy - 2y) = -5y^2 \\
\frac{dy}{dx} = \frac{-5y^2}{10xy - 2y} = \frac{-5y}{10x - 2} \\
\]

2 pts for finish

-1 pt if write \(\frac{dy}{dx}\) =...

-1 pt if write \(\frac{dy}{dx}\) =...

(b) (5 points) Find the equation of the tangent line to the graph at the point \(x = 0\).

\[
\gamma - y_0 = m(x - x_0) \\
\]

when \(x = 0\), \(y = \ldots\)

\(5(0)y^2 - y^2 = 5\)

\(-y^2 = 5\)

\(y^2 = -5\)

\(-1\) if \(y = \pm \sqrt{5}\)

2 pts to get to here

\(y\) is undefined

\(x = 0\) is not a point on the graph

\(+1\) for line \(y = \pm \sqrt{5}\)

\(\frac{dy}{dx} = \frac{-25}{2\sqrt{5}}\) +1 if this

\(+2\) if plug in

2
(2) (10 points) A company has found that its weekly profit from the sale of $x$ auto engines is given by:

$$P(x) = -x^2 + 100x + 300$$

The profit, $P(x)$, is given in hundreds of dollars. Production bottlenecks limit the number of engines that can be made per week to no more than 40, while a long-term contract requires that at least 10 engines be made each week. Find the maximum possible weekly profit that the firm can make.

$$P'(x) = -2x + 100$$

Critical Points:

0 = -2x + 100 (i)

2x = 100
x = 50

(3) (10 points) Use the definite integral to find the area between the x-axis and the curve for the function $f(x) = x^2 - 6x + 8$ on the interval $[3, 5]$.

$$X = \text{int} \ (y = 0)$$

0 = $x^2 - 6x + 8$

0 = (x-4)(x-2)

x = 4, 2

Only $x=4$ is in the interval $[3, 5]$

Area = $\int_3^4 (x^2 - 6x + 8) \, dx + \int_4^5 (x^2 - 6x + 8) \, dx$

$$= \left[ \frac{1}{3}x^3 - 3x^2 + 8x \right]_3^4 + \left[ \frac{1}{3}x^3 - 3x^2 + 8x \right]_4^5$$

$$= \left( \frac{1}{3}(4)^3 - 3(4)^2 + 8(4) \right) - \left( \frac{1}{3}(3)^3 - 3(3)^2 + 8(3) \right)$$

$$= \left( \frac{1}{3}(64) - 3(16) + 32 \right) - \left( \frac{1}{3}(27) - 3(9) + 24 \right)$$

$$= \left( \frac{64}{3} - 48 + 32 \right) - \left( 9 - 27 + 24 \right)$$

$$= \frac{1}{3} + 11 \frac{2}{3}$$

(2) Final Answer: $\frac{1}{3} + 11 \frac{2}{3}$
(2) (10 points) A company has found that its weekly profit from the sale of $x$ units of an auto part is given by:

$$P(x) = -x^2 + 80x + 300$$

The profit, $P(x)$, is given in thousands of dollars. Production bottlenecks limit the number of engines that can be made per week to no more than 30, while a long-term contract requires that at least 5 engines be made each week. Find the maximum possible weekly profit that the firm can make.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>675</td>
</tr>
<tr>
<td>30</td>
<td>1800</td>
</tr>
</tbody>
</table>

$$180,000 \text{ at } x=3$$

(3) (10 points) Use the definite integral to find the area between the $x$-axis and the curve for the function $f(x) = x^2 - 6x + 8$ on the interval $[3, 5]$. 

(same)
(4) **(20 points)** Your own a factory which manufactures bottles of wine from your vineyard. Over the years, you see a demand for about 1,000 bottles of your most expensive wine, Snobee Graip (it's French). It costs $5 to make 1 bottle of wine, $2 to store 1 bottle of wine for a year, and $500 to set up the factory for production.

(a) **(10 points)** Assuming you calculate your storage costs by the average number of items in storage, find the number of wine bottles that should be produced in each lot throughout the year in order to minimize total costs.

\[
\text{Total Cost} = \text{Production Cost} + \text{Storage Cost}
\]

\[
= (500 + 5q) \left( \frac{1000}{q} \right) + (2)q
\]

\[
(2) \quad T_c(q) = 500000q^{-1} + 5000 + q
\]

\[
(2) \quad T_c'(q) = -500000q^{-2} + 1
\]

\[
0 = -\frac{500000}{q^2} = 1
\]

\[
500000 = q^2
\]

\[
q = 707.12
\]

(2) \quad \text{They should produce 707 bottles}

(b) **(10 points)** Your storage situation has changed, and you must now calculate storage costs by the maximum number of items in storage at a given time (not the average). Find the number of wine bottles that should be produced in each lot throughout the year to minimize total costs.

\[
T_c(q) = (500 + 5q) \left( \frac{1000}{q} \right) + 2q
\]

(2) \quad = \frac{500000}{q} + 5000 + 2q

(2) \quad T_c'(q) = -500000q^{-2} + 2

(2) \quad 0 = -\frac{500000}{q^2} + 2

\[
\frac{500000}{q^2} = 2
\]

\[
q = 2500 = q^2
\]

\[
q = \sqrt{2500} = 500
\]

They should produce 500 bottles in a lot
(4) **(20 points)** Your own a factory which manufactures bottles of wine from your vineyard. Over the years, you see a demand for about 1,000 bottles of your most expensive wine, Snobee Graip (it's French). It costs $5 to make 1 bottle of the wine, $2 to store 1 bottle of wine for a year, and $500 to set up the factory for production.

(a) **(10 points)** Assuming you calculate your storage costs by the average number of items in storage, find the number of wine bottles that should be produced in each lot throughout the year in order to minimize total costs.

\[
q = \sqrt{\frac{2fM}{k}}
\]

\[
f = 500 \quad (2) \\
M = 1,000 \quad (2) \\
k = 8\,2 \quad (2)
\]

\[
= 707.12
\]

\[
\text{Check in total cost}
\]

\[
\text{or} \quad 707 \text{ or } 708 \quad (\text{whichever minimize, total cost})
\]

Produce **707 bottles in 1 lot** \( (1) \)

(b) **(10 points)** Your storage situation has changed, and you must now calculate storage costs by the maximum number of items in storage at a given time (not the average). Find the number of wine bottles that should be produced in each lot throughout the year to minimize total costs.

\[
M = 50
\]

\[
k = 1
\]

\[
f = 20
\]
(4) (20 points) Your own a factory which manufactures bottles of wine from your vineyard. Over the years, you see a demand for about 50 bottles of your most expensive wine, Snobee Grap (it's French). It costs $10 to make 1 bottle of the wine, $1 to store 1 bottle of wine for a year, and $200 to set up the factory for production.

(a) (10 points) Assuming you calculate your storage costs by the average number of items in storage, find the number of wine bottles that should be produced in each lot throughout the year in order to minimize total costs. \[
\text{Total Cost} = TC(q) = (200 + 10q) \left( \frac{50}{q} \right) + (1) \left( \frac{q^2}{2} \right) \]
\[
TC(q) = 10000q^{-1} + 500 + \frac{1}{2} q^2 \]
\[
TC'(q) = -10000q^{-2} + \frac{1}{2} \frac{q}{q^2} \]
\[
0 = -10000 \frac{1}{q^2} + \frac{1}{2} \]
\[
\frac{16}{q^2} = \frac{1}{2} \]
\[
16 = q^2 \]
\[
q = \sqrt{16} = 4 \]

(b) (10 points) Your storage situation has changed, and you must now calculate storage costs by the maximum number of items in storage at a given time (not the average). Find the number of wine bottles that should be produced in each lot throughout the year to minimize total costs. \[
\text{Total Cost} = TC(q) = (200 + 10q) \left( \frac{50}{q} \right) + q \]
\[
TC(q) = 10000q^{-1} + 500 + q \]
\[
TC'(q) = -10000q^{-2} + 1 \]
\[
0 = -10000 \frac{1}{q^2} + 1 \]
\[
10000 = \frac{1}{q^2} \]
\[
q = \sqrt{10000} = 100 \]
\[
100 = q \]
(5) (20 points) Suppose the demand function for a product is given by
\[ q = D(p) = 735 - 5p^2 \]

(a) (10 points) The company currently sells their product for $8 and are thinking of increasing their price. What do you recommend to them? Would a small increase in price increase their revenue, or would sales drop off at a higher price, thus decreasing their revenue? Justify your answer with mathematics.

\[ E = -\frac{p}{q} \frac{dq}{dp} \]

Knowing the correct formula when \( p = 8 \)

\[ E = \frac{10(8)^2}{735 - 5(8)^2} = 1.5422 \]

\( E > 1 \) So at this price demand is elastic

So a small % change in price leads to a larger % change in demand, and total revenue will decrease as price increases.

I would not recommend that they increase the price.

(b) (10 points) What price \( p \) should be charged to maximize revenue?

Revenue is maximized when \( E = 1 \)

\[ \frac{10p^2}{735 - 5p^2} = 1 \]

\[ 10p^2 = 735 - 5p^2 \]

\[ 15p^2 = 735 \]

\[ p^2 = 49 \]

\[ p = 7 \]

They should charge $7 to maximize revenue.

Final Answer
(5) (20 points) Suppose the demand function for a product is given by

\[ q = D(p) = 300 - 10p^2 \]

(a) (10 points) The company currently sells their product for $6 and are thinking of increasing their price. Would you recommend this course of action? Justify your answer with mathematics.

\[ E = - \frac{p}{q} \cdot \frac{dq}{dp} \]

\[ q = 300 - 10p^2 \]

\[ \frac{dq}{dp} = -20p \] (I) Derivative

\[ E = - \frac{p}{300-10p^2} \cdot (-20p) = \frac{20p^2}{300-10p^2} \] (I)

When \( p = 6 \)

\[ E = \frac{20(6)^2}{300-10(6)^2} = \frac{720}{-60} = -12 \]

\( E < 1 \).

So at this price demand is inelastic.

I would recommend increasing price.

(b) (10 points) What price \( p \) should be charged to maximize revenue?

Revenue is maximized when \( E = 1 \) (5)

\[ \frac{20p^2}{300-10p^2} = 1 \]

\[ 20p^2 = 300 - 10p^2 \]

\[ 30p^2 = 300 \]

\[ p^2 = 10 \]

\[ p = \sqrt{10} \]

\[ p = \$3.16 \] (2) Final answer

They should charge \$3.16 to maximize revenue.
(6) (10 points) A worker is constructing a cubical box that must contain 100 \( ft^3 \), with an error of no more than 0.1 \( ft^3 \). How much error can be tolerated in the length of each side of the box to ensure the volume is within tolerance? (i.e. the volume of the box is within the amount specified).

\[
V = x^3 \quad (2) \quad (\text{Volume Equation})
\]

\[
V'(x) = 3x^2 \quad (2) \quad \text{derivative}
\]

\[
dV \approx V'(x) \, dx
\]

\[
0.1 = 3 \left( \frac{3}{100} \right)^2 \, dx \quad (2)
\]

\[
\frac{0.1}{3 \left( \frac{3}{100} \right)^2} \approx dx \quad \approx 0.00718 \, ft
\]

(1 pt) For solving \( \cdot 100 \pm 0.1 \Rightarrow x^3 \)

(7) (10 points) Find the indefinite integral:

\[
\int (t^2 - 2)\sqrt{t^3 - 6t + 1} \, dt
\]

\[
\begin{align*}
\text{Let } u &= t^3 - 6t + 1 \quad (2) \\
\text{Then } du &= (3t^2 - 6) \, dt \quad (1) \\
\int (t^2 - 2)\sqrt{u} \, dt &= \int u^{1/2} (t^2 - 2) \, dt \\
&= \int u^{1/2} (t^2 - 2) \, dt \\
&= \int u^{1/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \int u^{3/2} \, du \\
&= \frac{1}{3} \left[ \frac{2}{5} u^{5/2} \right] + C = \frac{2}{15} (t^3 - 6t + 1)^{3/2} + C
\end{align*}
\]
(6) **(10 points)** A worker is cutting a square from a piece of metal. The specifications call for an area of 25 $ft^2$ with an error of no more than 0.2 $ft^2$. How much error can be tolerated in the length of each side of the square to ensure the area is within tolerance? (i.e. the area of the square is within the amount specified).

\[
A(x) = x^2 \quad (2)
\]

\[
A'(x) = 2x \quad (2)
\]

\[
dA \approx A'(x) \, dx \quad (2)
\]

\[
d(2) \approx A'(5) \, dx
\]

\[
\frac{2}{10} \approx dx \approx 0.02 \quad (2)
\]

\(
\) (Note for approximate instead of errors)

(7) **(10 points)** Find the indefinite integral:

\[
\int (t^2 - 2) \sqrt{t^3 - 6t + 1} \, dt
\]

\[
= \int \ln(t^2 - 2) \, dt
\]

\[
= \int \frac{1}{2} \ln(t^2 - 2) \, dt
\]

\[
= \frac{1}{3} \ln(t^2 - 2) + C
\]

\[
= \frac{2}{9} (t^3 - 6t + 1)^{\frac{3}{2}} + C
\]

\(13 \text{ pts}^+ \text{ for only correct answers} \)
(8) (10 points) The change in height of a tree is measured once every 5 years on March 1st. A representative from the forest preserve tells you that the tree was planted 20 years ago (in 1995) when it was 8 feet tall. You've organized the data given to you in the table below. \( t \) is given in years since 1995, and \( h'(t) \) is in feet.

<table>
<thead>
<tr>
<th>( t ) (time)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h'(t) ) (change in height)</td>
<td>14.5</td>
<td>57</td>
<td>155</td>
<td>336</td>
</tr>
</tbody>
</table>

You construct the curve of best fit through your data points and obtain the following formula which describes the change in height of the tree:

\[
h'(t) = 2 + \left( \frac{1}{9} \right) t^2
\]

(a) (5 points) Using calculus, write a formula, \( h(t) \), for the height of the tree at time \( t \).

\[
h(t) = \int h'(t) \, dt = \int \left( 2 + \frac{1}{9} t^2 \right) \, dt
\]

\[
= 2t + \frac{1}{3} \cdot \frac{1}{9} t^3 + C
\]

\[
h(t) = 2t + \frac{1}{27} t^3 + C
\]

\( t = 0, \ h(0) = g(4) \)

\[
g = 2(0) + \frac{1}{27} (0)^3 + C
\]

\[
g = C
\]

\[
h(t) = 2t + \frac{1}{27} t^3 + 8
\]

(b) (5 points) What do you predict the height of the tree will be in 2020? Show all work.

\[
h(25) = a(25) + \frac{1}{27} (25)^3 + 8
\]

\[
= 50 + \frac{1}{27} \cdot (15625) + 8
\]

\[
= 636.7 \text{ feet tall}
\]

I predict the tree will be approx. 636.7 feet tall.