(1) (20 points) Consider the following table of values of the functions $f$ and $g$ and their derivatives at various points:

$$
\begin{array}{c|cccc}
 x & 1 & 2 & 3 & 4 \\
 f(x) & 3 & 4 & 2 & 1 \\
 f'(x) & -5 & -6 & -7 & -11 \\
 g(x) & 4 & 1 & 2 & 3 \\
 g'(x) & 2 & 3 & 4 & 6 \\
\end{array}
$$

Use the table above to find the indicated derivatives. If the table does not contain the information necessary to answer the question, write "Not enough information" and EXPLAIN what information you would need provided to complete the question.

(a) (5 points) Find $h'(1)$ when $h(x) = f(x)g(2x - 1)$

$$
\begin{align*}
h'(x) &= f(x) \cdot (g(2x-1))' + g(2x-1) \cdot f'(x) \\
h'(1) &= f(1) \cdot g'(1) \cdot 2 + g(1) \cdot f'(1) \\
      &= (3)(-5) + (4)(-11) \\
      &= -8
\end{align*}
$$

(b) (5 points) Find $h'(2)$ when $h(x) = e^{f(x)}$

$$
\begin{align*}
h'(x) &= e^{f(x)} \cdot f'(x) \\
h'(2) &= e^{f(2)} \cdot f'(2) \\
      &= e^4 \cdot (-6) \\
      &= -6e^4
\end{align*}
$$

(c) (5 points) Find $h'(1)$ when $h(x) = \frac{\ln(x)}{g(x)}$

$$
\begin{align*}
h'(x) &= \left[ \frac{g(x) \left( \frac{1}{x} \right) - \ln(x) \cdot g'(x)}{(g(x))^2} \right] \\
h'(1) &= g(1) - \frac{\ln(1) \cdot g'(1)}{(g(1))^2} \\
      &= 4 - \frac{1}{16} \\
      &= \frac{15}{16}
\end{align*}
$$

(d) (5 points) Write the equation of the tangent line to $h(x)$ at $x = 3$ if $h(x) = f(x) + g(x) + 2$

$$
\begin{align*}
h'(x) &= f'(x) + g'(x) \\
h'(3) &= f'(3) + g'(3) \\
      &= (-7) + 4 \\
      &= -3 \\
\end{align*}
$$

$$
\begin{align*}
h(3) &= f(3) + g(3) + 2 \\
      &= 2 + 9 + 2 \\
      &= 13
\end{align*}
$$

$$
\begin{align*}
m &= \text{slope} = h'(3) = -3 \\
y - y_0 &= m(x - x_0) \\
(3, 6) &= \text{point} \\
m &= -3 \\
2 &= -3(x - 3) \\
6 &= -3(x - 3) \\
7 &= -3 \cdot 2 \\
7 &= -6
\end{align*}
$$
(1) (20 points) Consider the following table of values of the functions $f$ and $g$ and their derivatives at various points:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>-5</td>
<td>-6</td>
<td>-7</td>
<td>-11</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$g'(x)$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Use the table above to find the indicated derivatives. If the table does not contain the information necessary to answer the question, write "Not enough information" and EXPLAIN what information you would need provided to complete the question.

(a) (5 points) Find $h'(1)$ when $h(x) = f(x)g(3x - 1)$

$$h'(x) = f(x)g'(3x - 1) + g(3x - 1)f'(x)$$

$$h'(1) = f(1)g'(2) + g(2)f'(1)$$

$$= 1 - 3 + 1 - 5 = -4$$

(b) (5 points) Find $h'(2)$ when $h(x) = e^{f(x)}$

(c) (5 points) Find $h'(1)$ when $h(x) = \frac{ln(x)}{g(x)}$

(d) (5 points) Write the equation of the tangent line to $h(x)$ at $x = 2$ if $h(x) = f(x) + g(x) + 2$.

$$h(2) = f(2) + g(2) + 2$$

$$= 4 + 1 + 2 = 7$$

$$y - 7 = -3(x - 2)$$
Problem 2 (20 points) An art gallery orders 50 prints by a famous artist. If each print in the limited edition is priced at \( p \) dollars, it is expected that \( q = 500 - 5p \) prints will be sold. (This is the demand for the item at a particular price \( p \)).

(a) (2 points) Use the information given to deduce practical limitations on the values of \( p \).
- \( 0 \leq p \leq 100 \) (no negative price)
- \( 0 \leq q \leq 50 \) (no negative quantity)
- \( 0 \leq s \leq 500 - 5p \) (supply)
- \( 0 \leq p \leq 100 \) (price)

(b) (4 points) Write an equation for the revenue \( R(p) \), the revenue taken in when selling each painting at a price \( p \).
\[
R(p) = \left[ \frac{q}{100} \right] \cdot (500 - 5p)
\]
\[
R(p) = \frac{(500 - 5p)}{100}
\]

(c) (6 points) Find the marginal revenue when you have sold 1 painting. Explain your answer in a complete sentence.
\[
R'(p) = \frac{-5}{100} = -0.05
\]

(d) (8 points) If you were the owner of the gallery, what price would you charge for each print? Explain the reasoning behind your decision.

\[
\text{Maximize Revenue}
\]
\[
R'(p) = 0
\]
\[
500 - 10p = 0
\]
\[
500 = 10p
\]

I would charge \$50 for each print b/c this is the price that maximizes revenue.
(2) (20 points) An art gallery orders 50 prints by a famous artist. If each print in the limited edition is priced at p dollars, it is expected that q = 500 - 5p prints will be sold. (This is the demand for the item at a particular price p).

(a) (2 points) Use the information given to deduce practical limitations on the values of p.

(b) (4 points) Write an equation for the revenue $R(p)$, the revenue taken in when selling each painting at a price $p$.

(c) (6 points) Find the marginal revenue when you have sold 2 paintings. Explain your answer in a complete sentence.

\[ R'(p) = 500 - 10p \]
\[ R'(2) = 500 - 20 = 480 \]

When you’ve sold 2 paintings, your revenue is increasing by $480 per painting.

(d) (8 points) If you were the owner of the gallery, what price would you charge for each print? Explain the reasoning behind your decision.

\[ p = \frac{500 - q}{5} \]
\[ R(q) = \frac{1}{5}(500q - q^2) \]
\[ R'(q) = 100 - \frac{2q}{5} \]
\[ R'(1) = 100 - \frac{2}{5} = \frac{498}{5} \]
\[ R'(2) = 100 - \frac{4}{5} = \frac{496}{5} \]
(3) (20 points) After losing their Coruscant factory to the Galactic Empire, Teddy's Total Destruction has found their weapon sales have fallen considerably. They are considering increasing their advertising budget in the hopes of boosting sales. A small pilot study estimates that the monthly sales function, \( f(x) \)

\[ f(x) = -5x^4 + 10x^3 \]

represents the number of plasma blasters sold each month and \( x \) is the number of advertising circulars dropped from the Millennium Falcon (in thousands).

(a) (10 points) What level of advertising will maximize sales of plasma blasters?
Write out the answer as a complete sentence.

(b) (10 points) Does the advertising campaign have a point of diminishing returns?
If yes, what is it? Write out the answer as a complete sentence.
(4) (20 points) You work for a company that manufactures large engines for cruise ships and planes. The weekly cost of manufacturing \( q \) units is

\[ C(q) = 5q^3 - 3q + 1000 \]

where \( C(q) \) is in tens of thousands of dollars. Currently your company is manufacturing 5 engines a week.

(a) (5 points) ESTIMATE the cost of producing the 6th engine.

\[ C'(5) = 15(5)^2 - 3 = 375 - 3 = 372 \]

I estimate that the cost to produce the 6th engine is $3,720,000 dollars.

(b) (5 points) Check your result by computing the actual cost of producing the 6th engine.

\[ \left( \frac{C(6) - C(5)}{6 - 5} \right) = \frac{(5(6)^3 - 3(6) + 1000) - (5(5)^3 - 3(5) + 1000)}{1} = \frac{4520}{1} = 4520 \]

The cost to produce the 6th engine is $45,200,000 dollars.

(c) (10 points) Calculate the marginal average cost when producing 5 engines. Explain the number you have found in words.

Average cost

\[ \bar{C}(x) = \frac{C(x)}{x} \]

Marginal average cost

\[ \left( \frac{C(x)}{x} \right)' = \frac{(C(x))'}{x} \] (2 pts)

\[ \left( \frac{C(5)}{5} \right)' = \frac{5 \cdot C'(5) - C(5)}{25} \] (3 pts)

\[ \frac{5 \cdot 372 - 1610}{25} = 10 \]

\[ \frac{5 \cdot 372 - 1610}{10} = 10 \]
(4) (20 points) You work for a company that manufactures large engines for cruise ships planes. The weekly cost of manufacturing \( q \) units is

\[
C(q) = 5q^2 - 3q + 1000
\]

where \( C(q) \) is in tens of thousands of dollars. Currently your company is manufacturing 5 engines a week.

(a) (5 points) ESTIMATE the cost of producing the 5th engine.

\[
C'(q) = 15q - 3
\]

\[
C'(4) = 15(4) - 3 = 57
\]

I estimate that the cost to produce the 5th engine is 2,370,000 dollars.

(b) (5 points) Check your result by computing the actual cost of producing the 5th engine.

\[
C(5) - C(4) = 3029000
\]

\[
= 1610 - 1308 = 302
\]

(c) (10 points) Calculate the marginal average cost when producing 4 engines. Explain the number you have found in words.

\[
\left( \frac{C(q)}{q} \right)' = \frac{XC'(x) - C(x)}{x^2}
\]

\[
= 4 \cdot C'(4) - C(4)
\]

\[
= 4 \cdot 57 - 1308 = -22.5
\]

\[
= 2250000
\]
(5) (5 points) Twoflower and Rincewind check into the “Evermore Hotel” in Discworld, where, every evening, the owners raise the price to rent a room for the night. The current rate to rent a room for the night can be approximated by $f(t)$ where $t$ is the number of nights since the beginning of the calendar year. Find the value of $t$ when the rent charged for the room is increasing most rapidly. Explain the reasoning behind your answer. Below are the graphs of the function $f(t)$, $f'(t)$, and $f''(t)$.

The rent charged for the room is increasing most rapidly when $t = 25$ (25 days since the beginning of the calendar year).

Where rent is increasing most rapidly = Maximum rate of change of rent = Maximum of $f'(x)$

-2 if they consider $f''(t)$ instead of $f(t)$

-1 For incorrect facts

For absolutely zero points, name the sci-fi author in whose honour this question was written.

$f(t) = 25$
$f'(t) = 25$

I love Terri Pratchett

-1 More detail
(6) (5 points) Is one of the graphs below the graph of the following function? Show all work ON THE NEXT PAGE to justify your answer and receive full credit. If, according to your work, none graphs match the given function, sketch a graph of what the function should look like.

\[ f(x) = \frac{3x^2 + 4x + 1}{(x-1)(x+2)} = \frac{3x^2 + 4x + 1}{x^2 + x - 2} \]

Graph must match work to earn this point.

If work does not completely determine graph, still give point.
(6) (5 points) Is one of the graphs below the graph of the following function? Show all work ON THE NEXT PAGE to justify your answer and receive full credit. If, according to your work, none graphs match the given function, sketch a graph of what the function should look like.

\[ f(x) = \frac{3x^2 + 4x + 1}{(x - 1)(x + 2)} \]
**YOU MAY SHOW WORK FOR PROBLEM 6 HERE**

**x-intercepts (y = 0)**

\[ 0 = 3x^2 + 4x + 1 \]
\[ 0 = (3x+1)(x+1) \]
\[ 3x+1 = 0 \quad x+1 = 0 \]
\[ x = -\frac{1}{3} \quad x = -1 \]

\[ (-\frac{1}{3}, 0) \quad (-1, 0) \]

**y-intercepts (x = 0)**

\[ y = \frac{1}{(-1)(2)} = -\frac{1}{2} \]

\[ (0, -\frac{1}{2}) \]

This is enough work to identify the graph.

**VA**

\[ \frac{(x-1)(x+2)}{0} \]

\[ x = 1, \quad x = -2 \]

**HA**

\[ \lim_{x \to +\infty} \frac{3x^2 + 4x + 1}{x^2 + x - 2} = \frac{(3 + \frac{4}{1} + \frac{1}{x})}{1 + \frac{1}{x} - \frac{2}{x^2}} \cdot \frac{x^2}{x^2} = 3 \]

\[ \lim_{x \to -\infty} \frac{3 + \frac{4}{1} + \frac{1}{x}}{1 + \frac{1}{x} - \frac{2}{x^2}} \cdot \frac{x^2}{x^2} = 3 \]

\[ y = 3 \]

(1 pt)

(*4 pts*)

For some sort of relevant work

\(\text{consider } f'(x) \text{ instead of } f(x)\)
(7) **(2 points)** Which of the graphs above has a point of diminishing returns?

(a) Only Graph A

(b) Only Graph B

(c) Both Graph A and B have points of diminishing returns.

(d) Neither Graph A nor B has a point of diminishing returns.

(8) **(8 points)** Next to each of the curves below, write any/all letters that appropriately describe the curve.

(a) $f(x)$ is increasing  
(b) $f(x)$ is decreasing  
(c) $f'(x)$ is positive  
(d) $f'(x)$ is negative  
(e) $f'(x)$ is increasing  
(f) $f'(x)$ is decreasing  
(g) $f''(x)$ is positive  
(h) $f''(x)$ is negative
(7) (2 points) Which of the graphs above DOES NOT have a point of diminishing returns?

(a) Only Graph A

(b) Only Graph B

(c) Both Graph A and B have points of diminishing returns.

(d) Neither Graph A nor B has a point of diminishing returns.

(8) (8 points) Next to each of the curves below, write any/all letters that appropriately describe the curve.

(a) $f(x)$ is increasing  
(b) $f(x)$ is decreasing  
(c) $f'(x)$ is positive

(d) $f'(x)$ is negative  
(e) $f'(x)$ is increasing  
(f) $f'(x)$ is decreasing

(g) $f''(x)$ is positive  
(h) $f''(x)$ is negative