If they have bad notation anywhere on #1, (1) just once
If they have bad notation anywhere on #2, (1) just once

1. (20 points) (5 points each) Evaluate the following limits in (a) - (d). Show sufficient justification for each answer. If the limit does not exist, explicitly state "does not exist". For infinite limits you must state if it is $\infty$ or $-\infty$. Do not use L'Hopital's Rule.

(a) \[ \lim_{x \to \infty} \frac{\sqrt{x^2 + 4}}{4x + 1} = \lim_{x \to \infty} \frac{x^2(1 + \frac{4}{x^2})}{(4x + 1)^2} = \lim_{x \to \infty} \frac{\sqrt{x^2 + 4}}{4x + 1} \]

(b) \[ \lim_{x \to -\infty} \frac{x + 1}{x^2 - 3x - 10} = \lim_{x \to -\infty} \frac{x + 1}{(x + 2)(x - 5)} \]

(c) \[ \lim_{x \to \infty} \sqrt{9x^2 + 4x + 1} - 3x = \infty - 3x \]

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2. (10 points) (a) State the limit definition of the derivative of $f(x)$ as a function.

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

(b) Using the limit definition of the derivative show the derivative of $f(x) = 2x^2 - 3x$ is $f'(x) = 4x - 3$.

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{[2(x+h)^2 - 3(x+h)] - [2x^2 - 3x]}{h} \\
         &= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 3x - 3h}{h} - \frac{2x^2 - 3x}{h} \\
         &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h}{h} - \frac{2x^2 - 3x}{h} \\
         &= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} \\
         &= \lim_{h \to 0} (4x - 3) \\
         &= 4x - 3
\end{align*}
\]
If they have incorrect parenthesis in #3, each time.

Same for #4

3. (25 points) (5 points each) For each of the functions below, compute the derivative with respect to the indicated variable, using any methods we have covered in class. You do not need to simplify your answers.

(a) \( f(x) = \cos(\pi^2) \ln(\pi) 3e^x \)

\[ f'(x) = 0 \]

(b) \( f(x) = \frac{\arcsin(x)}{x^4 + 1} \)

\[ f'(x) = \left( x^4 + 1 \right) \left( \frac{1}{\sqrt{1-x^2}} \right) - \arcsin(x) \left( 4x^3 \right) \]

(c) \( y = (e + 6e^{2x^2})^{\frac{1}{2}} \)

\[ y' = \frac{1}{a} \left( e + 6e^{2x^2} \right) \left( 0 + 6e^{2x^2} \right) \]

\(-1\) for extra terms
(d) \[ y = (\cos(x))^{\cos(x)} \]

\[ \ln(y) = \ln((\cos(x))^{\cos(x)}) \]

\[ \ln(y) = \cos(x) \cdot \ln(\cos(x)) \]

\[ \frac{1}{y} \cdot y' = \cos(x) \cdot \frac{1}{\cos(x)} (-\sin(x)) + \ln(\cos(x)) \cdot (-\sin(x)) \]

\[ \frac{1}{y} \cdot y' = \cos(x) - \ln(\cos(x)) \cdot \frac{\sin(x)}{\cos(x)} \]

(e) Find \( \frac{dy}{dx} \) when \( x^2 - 4xy = 2y + 4 \)

\[ 2x - (4x \frac{dy}{dx} + y) \frac{dx}{dy} = 2 \frac{dy}{dx} \]

\[ 2x - 4x \frac{dy}{dx} = 2 \frac{dy}{dx} \]

\[ 2x = 4x \frac{dy}{dx} + 2 \frac{dy}{dx} \]

\[ \frac{dy}{dx} = \frac{2x - 4y}{4x + 2} \]

\[ \text{Attempts to take derivative of both sides implicitly} \]

4. (7 points) Find the equation of the tangent line to the curve \( y = \tan\theta \sec\theta \) when \( (\theta, y) = (\frac{\pi}{4}, \sqrt{2}) \).

\[ y' = \tan\theta (\sec\theta \tan\theta) + \sec\theta \sec^3\theta \]

\[ = \sec^2\theta + \sec^3\theta \]

\[ = \frac{\sin^2\theta + 1}{\cos^3\theta} \]

\[ = \frac{1}{\cos^3\theta} \]

\[ = \left( \frac{\sqrt{2}}{2} \right)^3 + \frac{1}{\left( \frac{\sqrt{2}}{2} \right)^3} \]

\[ = \frac{3}{\sqrt{2}} + \frac{\sqrt{2}}{3} \]

\[ = \frac{12}{3\sqrt{2}} = \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \]

\[ y' \left( \frac{\pi}{4} \right) = \frac{3\sqrt{2}}{2} \]

\[ \tan \left( \frac{\pi}{4} \right) \sec \left( \frac{\pi}{4} \right) \text{ is ok} \]

\[ y - \sqrt{2} = 3 \sqrt{2} \left( \frac{\pi}{4} - \frac{\pi}{4} \right) \]

\[ \frac{1}{2} \text{ for missing ( )} \]

\[ \left( \sec \left( \frac{\pi}{4} \right) \tan^2 \left( \frac{\pi}{4} \right) + \sec^3 \left( \frac{\pi}{4} \right) \right) \text{ ok} \]
5. (8 points) Find the equation of each horizontal asymptote for the given function.

\[ f(x) = \frac{10 - e^{2x}}{e^{4x} - 6} \]

\[
\lim_{x \to +\infty} \frac{10 - e^{2x}}{e^{4x} - 6} = \lim_{x \to +\infty} \frac{10/e^{2x} - e^{2x}}{e^{4x}/e^{2x} - e^{2x}} = \frac{0}{1} = 0
\]

\[
\lim_{x \to -\infty} \frac{10 - e^{2x}}{e^{4x} - 6} = -\frac{10}{6} = -\frac{5}{3}
\]

\[ y = 0, \quad y = -\frac{5}{3} \]

6. (8 points) (3/5) This problem is concerned with the Intermediate Value Theorem.

(a) State the Intermediate Value Theorem.

If \( f(x) \) is continuous on \( [a, b] \) and if \( N \) is a number \( f(a) < N < f(b) \), then there exists a number \( c \) in \( (a, b) \) so \( f(c) = N \)

(b) Show that \( f(x) = x^3 - x + 3 \) has a root in the interval \( (-10, -1) \).

\( f(x) \) is a polynomial, so it is continuous everywhere

\[ f(-10) = -\text{#} \]

\[ f(-1) = +\text{#} \]

So by the IVT, there is a \( \# c \) in \( (-10, -1) \) where \( f(c) = 0 \)

"Some true conclusion"
5. (8 points) Find the equation of each horizontal asymptote for the given function.

\[ f(x) = \frac{10 - e^{2x}}{e^{3x} - 6} \]

Set up:

\[ \lim_{x \to \infty} \frac{10 - e^{2x}}{e^{3x} - 6} = \lim_{x \to \infty} \frac{10 - e^{2x}}{e^{3x} - 6} \]

Correct answer:

\[ \lim_{x \to \infty} \frac{10 - e^{2x}}{e^{3x} - 6} = 0 \]

(Eqns)

\[ y = 0, \quad y = -5/3 \]

6. (8 points) (3/5) This problem is concerned with the Intermediate Value Theorem.

(a) State the Intermediate Value Theorem.

If \( f(x) \) is continuous on \((a, b)\) and if \( N \) is a number \( f(a) \leq N \leq f(b) \) then there exists a number \( c \) in \((a, b)\) so \( f(c) = N \).

(b) Show that \( f(x) = x^3 - x + 3 \) has a root in the interval \((-10, -1)\).

\( f(x) \) is a polynomial, so it is continuous everywhere.

\[ f(-10) = -17 \] 1pt

\[ f(-1) = 1 \] 1pt

So by the I VT, there is a \( c \) in \((-10, -1)\) where \( f(c) = 0 \).
7. (6 points) Suppose \( \lim_{x \to 1} f(x) = 0 \) and that \( f(x) \) is continuous at \( x = 1 \). Given that \( f(x + 2) \leq g(x) \leq 3f(x + 2) \) compute the limits below or state that there is not enough information. Be sure to refer to theorems by names and verify hypotheses, if applicable.

1. \( \lim_{x \to 1} g(x) \)
   
   \[
   \lim_{x \to 1} f(x+2) = f(1) = 0 \\
   \lim_{x \to 1} 3f(x+2) = 3f(1) = 0
   \]
   
   Since \( f(x) \) is continuous at \( x=1 \)
   
   \[
   \lim_{x \to 1} f(x) = f(1) = 0
   \]
   
   So by the Squeeze Theorem
   
   \[
   \lim_{x \to 1} g(x) = 0
   \]

2. \( \lim_{x \to 1} g(x) \)
   
   There is not enough info

8. (12 points) Use the given graph of \( f \) to state the value of each quantity, if it exists. (If an answer does not exist, enter DNE).

![Graph with limits](image)

- \( \lim_{x \to 2^-} f(x) \) 3
- \( \lim_{x \to 2^+} f(x) \) 1
- \( \lim_{x \to 2} f(x) \) DNE
- \( f(2) \) 3
- \( \lim_{x \to 4} f(x) \) 4
- \( f(4) \) DNE
7. (6 points) Suppose \( \lim_{x \to 1} f(x) = 0 \) and that \( f(x) \) is continuous at \( x = 1 \). Given that

\[
f(x + 2) \leq g(x) \leq 3f(x + 2)
\]

compute the limits below or state that there is not enough information. Be sure to refer to theorems by names and verify hypotheses, if applicable.

1. \( \lim_{x \to 1} g(x) \)

\[
\lim_{x \to 1} f(x + 2) = f(1) = 0 \quad 1pt
\]

\[
\lim_{x \to 1} 3f(x + 2) = 3f(1) = 0 \quad 1pt
\]

2. \( \lim_{x \to 1} g(x) \)

There is not enough information

8. (12 points) Use the given graph of \( f \) to state the value of each quantity, if it exists. (If an answer does not exist, enter DNE).

(a) \( \lim_{x \to 2^-} f(x) \)

3

(b) \( \lim_{x \to 2^+} f(x) \)

1

(c) \( \lim_{x \to 2} f(x) \)

DNE

(d) \( f(2) \)

3

(e) \( \lim_{x \to 4} f(x) \)

4

(f) \( f(4) \)

DNE
9. (2 points) The graph of \( f(x) \) is shown below. Which of the following could be the graph of the derivative function, \( f'(x) \)?

![Graph of \( f(x) \)](image)

A. ![Graph of \( f'(x) \)](image)
B. ![Graph of \( f'(x) \)](image)
C. ![Graph of \( f'(x) \)](image)
D. ![Graph of \( f'(x) \)](image)
E. ![Graph of \( f'(x) \)](image)

10. (2 points) **True** or False: If \( g(x) = x^5 \), then \( \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = 80 \).

\[
g'(x) = 5x^4 \\
g'(2) = 5(2)^4 \\
= 5(16) = 80
\]