To estimate a volume, we can split our figure into "n" many equal-sized pieces of width $\Delta x$, and sum up volumes as

$$V_{ol} = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i) \Delta x$$

$$= \int_{a}^{b} A(x) \, dx$$
Show that the volume of a sphere of radius \( r \) is
\[
V = \frac{4}{3} \pi r^3
\]
Place the sphere so the center is at the origin.
The cross-section at a vertical slice through \( x \) is a circle.
The radius of this circle is \( y \).
\[
x^2 + y^2 = r^2 \quad y = \sqrt{r^2 - x^2}
\]
Cross-sectional Area \( A = \pi y^2 = \pi (r^2 - x^2) \)
\[
V = \int_{-r}^{r} \pi (r^2 - x^2) \, dx = 2\pi \int_{0}^{r} \pi (r^2 - x^2) \, dx
\]
\[
= 2\pi \left[ r^2x - \frac{1}{3}x^3 \right]_{x=0}^{x=r}
\]
\[
= 2\pi \left[ r^3 - r^3/3 \right]
\]
\[
= \frac{4}{3} \pi r^3
\]
Consider the region between \( y = \sqrt{x} \) and the x-axis, over the interval \([1, 9]\). Set up integrals which calculate the volumes of the following solids.

a) Revolve this region around the x-axis

\[
Vol = \int_{1}^{9} \pi (\sqrt{x})^2 \, dx
\]

b) Revolve this region around the line \( y = 4 \).

Cross-sections: Washers

\[
A = \pi (r_{out}^2 - r_{in}^2)
\]

\[
r_{out} = 4, \quad r_{in} = (4 - \sqrt{x})
\]

\[
Vol = \pi \int_{1}^{9} (4)^2 - (4 - \sqrt{x})^2 \, dx
\]
c) Revolve this region around the y-axis.

Rotating a vertical cross-section sweeps out a cylindrical shell.

Rotating a horizontal cross-section sweeps out a washer.

We will integrate w.r.t. y.

**Cross-sections (horizontal)**

<table>
<thead>
<tr>
<th>Region</th>
<th>0 ≤ y ≤ 1</th>
<th>1 ≤ y ≤ 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{out}$</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>$r_{in}$</td>
<td>1</td>
<td>$x = y^2$</td>
</tr>
</tbody>
</table>

$$A = \pi (r_{out}^2 - r_{in}^2)$$

$$\text{Vol} = \int_{0}^{1} (9^2 - 1^2) \, dy + \int_{1}^{3} (9^2 - (y^2)^2) \, dy$$
d) Let \( R \) be the base of a solid for which cross-sections perpendicular to the \( x \)-axis are semi-circles.

Cross-sections

semi-circles

\[ A = \frac{1}{2} \pi r^2 \]

radius = \( \frac{1}{2}y \)

\[ = \frac{1}{2} \sqrt{x} \]

\[ V_{01} = \int_{1}^{9} \frac{1}{2} \pi \left( \frac{1}{2} \sqrt{x} \right)^2 \, dx \]
EX] A solid has a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

Set up an integral to center the base in the xy-plane

\[ x^2 + y^2 = 1 \]
\[ y^2 = 1 - x^2 \]

Cross-section (vertical)

Equilateral triangle

\[ \text{Area} = \frac{1}{2} \text{(base)} \text{(height)} \]

\[ h = \frac{\sqrt{3}}{2} s \]

\[ A = \frac{1}{2} s \left( \frac{\sqrt{3}}{2} s \right) = \frac{\sqrt{3}}{4} s^2 \]

In our drawing, \( s = 2y \)

\[ A = \frac{\sqrt{3}}{4} (2y)^2 = \frac{\sqrt{3}}{3} y^2 = \sqrt{3} (1-x^2) \]

Since we are taking a vertical slice, the area func must be w.r.t \( y \)

\[ \text{Vol} = \int_{-1}^{1} \sqrt{3} (1-x^2) \, dx \]

\[ \text{Vol} = \int_{-1}^{1} A(y) \, dy \]