Recall: **Fundamental Thm of Calculus**

If $f$ is continuous,

**Part 1** \[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

i.e. if $g(x) = \int_a^x f(t) \, dt$, then $g'(x) = f(x)$

**Part 2** \[ \int_a^b f(x) \, dx = F(b) - F(a) \]

(where $F'(x) = f(x)$)

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**Generalization of Part 1**

If \[ g(x) = \int_a^x f(t) \, dt \]

Then \[ g'(x) = f(x(a)) \cdot \frac{d}{dx}(x(a)) \]

\[ = f(qx) \cdot q'(x) \]
Example 1 \[ g(x) = \int \frac{x}{\cos(\sqrt{t^2})} \, dt \]

Then \[ g'(x) = \cos(\sqrt{x}) \]

Example 2 \[ g(x) = \int (1+t^3) \, dt \]

Then \[ g'(x) = -\int (1+t^3) \, dt \]

and \[ g'(x) = -(1+x^3) \]

Example 3 \[ F(x) = \int_{x}^{2x} \ln(t) \, dt \]

Then \[ F(x) = \int_{x}^{2x} \ln(t) \, dt = \int_{1}^{2} \ln(t) \, dt + \int_{1}^{2} \ln(t) \, dt = \int_{1}^{2} \ln(t) \, dt + \int_{1}^{2} \ln(t) \, dt \]

and \[ F'(x) = -\ln(x) + \ln(2x) - 2 \text{ (chain rule)} \]
Ex) \[ g(x) = \int_{e^2}^{x} e^t \, dt \quad \text{Then} \quad g'(x) = e^{x^2} \]

Ex) \[ g(x) = \int_{1}^{x^3} \sec^5(t) \, dt \quad \text{Then} \quad g'(x) = \sec^5(x^3) \cdot 3x \]

Ex) Find \[ \int (1 + \tan^2(x)) \, dx \]

\[ = \int \sec^2(x) \, dx = \tan(x) + C \]

Recall

\[ \sin^2 \theta + \cos^2 \theta = 1 \]

\[ \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \]

\[ \tan^2 \theta + 1 = \sec^2 \theta \]

Ex) Find \[ \int e^{ax} \, dx = \frac{1}{a} e^{ax} + C \]
Recall:

\[ (x^n)' = nx^{n-1} \]
\[ (e^{kx})' = ke^{kx} \]
\[ (5^x)' = 5^x \cdot \ln(5) \]

Derivatives that involve multiplying by a constant

So for antiderivatives we get:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( F(x) )</th>
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</thead>
<tbody>
<tr>
<td>( x^n )</td>
<td>( \frac{x^{n+1}}{n+1} )</td>
</tr>
<tr>
<td>( e^{kx} )</td>
<td>( \frac{1}{k} e^{kx} )</td>
</tr>
<tr>
<td>( a^x )</td>
<td>( \frac{1}{\ln(a)} a^x )</td>
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Anti-derivatives that involve dividing by a constant

**Example**

Find \( \int (5^x - 5^x) \, dx \)

\[
= \left[ \frac{5}{2} x^2 - \frac{5^x}{\ln(5)} \right]_0^1
\]

\[
= \left[ \frac{5}{2} (1)^2 - \frac{5^1}{\ln(5)} \right] - \left[ \frac{5}{2} (0)^2 - \frac{5^0}{\ln(5)} \right]
= \frac{5 - \frac{5}{\ln(5)}}{2} + \frac{1}{\ln(5)} \]
**Note**

\[ \int f(x) \, dx \quad \text{Indefinite Integral} \]
- anti-derivative + C
- infinite family of functions

\[ \int_{a}^{b} f(x) \, dx \quad \text{Definite Integral} \]
- signed area under curve
- a number calculated by Fundamental Theorem of Calculus (part 2)

**Ex**
Find the area bounded by y-axis, the line y = 6, \( y = 6^4\sqrt{x} \) using an integral in y

\[
\begin{align*}
\text{Area} &= \int_{0}^{6} \frac{y^4}{6^4} \, dy \\
&= \left[ \frac{1}{64} \cdot \frac{1}{5} y^5 \right]^{6}_{0} \\
&= \frac{1}{64} \cdot \frac{1}{5} (6^5 - 0) \\
&= \frac{6^5}{320} \\
&= \frac{6}{5}
\end{align*}
\]
Net Change Theorem (Really FTC0, Part 2)\[\int_{a}^{b} F'(x) \, dx = F(b) - F(a)\]

rate of change of $F$

if $F'$ is continuous on $[a,b]$.

Example: Suppose that a population is expected to increase at a rate of $6t^2 + 2$ people per year where $t =$ # of years from now. What is the expected change in population between $t = 1$ and $t = 3$?

Given: $P'(t) = 6t^2 + 2$

Want: $P(3) - P(1)$

\[
\int_{1}^{3} (6t^2 + 2) \, dt = \left[ 3t^2 + 2t \right]_{1}^{3} \]

\[
= \left[ 3(3)^2 + 2(3) \right] - \left[ 3(1)^2 + 2(1) \right] \]

\[
= 28 \text{ people} \]
Substitution Method

We had many techniques to take derivatives (product rule, quotient rule, chain rule) etc.

The substitution method is an integration technique that reverses a derivative taken with the chain rule.

Chain Rule \[ [f(g(x))]' = f'(g(x)) \cdot g'(x) \]

applies to taking derivatives of composite functions.

\[ [(5x^2+7)^9] = 9(5x^2+7)^8 \cdot (10x) \]
\[ = 90x(5x^2+7)^8 \]

Suppose you are given

\[ \int 90x(5x^2+7)^8 \, dx \]

How do you recognize the anti-derivative is \((5x^2+7)^9\)?
Notice

\[
[f(g(x))]' = f'(g(x)) \cdot g'(x)
\]

The result of a derivative taken with a chain always has one function plugged into another \((g(x) = u)\) with the derivative of that function multiplied on \((g'(x) = du)\)

\[= f'(u) \, du \]

Doing a change of variables simplifies the problem to focusing on finding the anti-derivative of \(f'(u)\)

**Ex:** \(\int 90x \cdot (5x^2 + 7)^8 \, dx\)

\(u = 5x^2 + 7\) \quad \text{choose an "inside" function to try as } u

\(du = 10x \, dx\) \quad \text{take the derivative of the "u" chosen, writing } dx

\(du = 90x \, dx\)

\(q \cdot du = 90x \, dx\)

\(= \int qu^8 \, du\) \quad \text{use the eqns of "u" and "du" to substitute the integrand}

\(= u^9 + C = (5x^2 + 7)^9 + C\)
\[ \int x^2 \left( \frac{3}{x^2 + 1} \right) \, dx = \int u^{3/4} \left( \frac{1}{3} \, du \right) \]

\[ u = x^2 + 1 \]
\[ du = 2x \, dx \]
\[ \frac{1}{3} \, du = \frac{1}{2} \, dx \]
\[ = \frac{1}{3} \left( x^2 + 1 \right)^{7/4} + C \]

\[ \int \frac{1}{3x + 10} \, dx = \int \frac{1}{u} \, du \]

\[ u = 3x + 10 \]
\[ du = 3 \, dx \]
\[ = \frac{1}{3} \ln |10u| + C \]

\[ \int \frac{\sin \left( \sqrt{x} \right)}{\sqrt{x}} \, dx = \int \sin(u) \cdot 2 \, du \]

\[ u = \sqrt{x} \]
\[ du = \frac{1}{2} \sqrt{x} \, dx \]
\[ = \frac{1}{2} \, dx \]
\[ = -2 \cos \left( \sqrt{x} \right) + C \]

Think about how substituting this integral as \[ \int \frac{\sin(u)}{u} \, dx \] is incorrect.
\[ \int \cos^4(\theta) \sin \theta \, d\theta = \int u^4 \, du \]

\[ u = \cos \theta \]
\[ du = -\sin \theta \, d\theta \]
\[ -du = \sin \theta \, d\theta \]
\[ = -\frac{1}{5} \cos^5 \theta + C \]

\[ \int e^{1/x} \frac{1}{x^2} \, dx \]

\[ u = \frac{1}{x} \]
\[ du = -\frac{1}{x^2} \, dx \]
\[ -du = \frac{1}{x} \, dx \]

\[ u = \frac{1}{x} \]
\[ u = 1 \]
\[ u = \frac{1}{2} \]

\[ = \int -e^u \, du = \left[-e^u\right]_1^{1/2} \]
\[ u = 1 \]
\[ = -e^{1/2} - \left[-e^1\right] \]
\[ = -e^{1/2} + e \]