Estimating Area under curve w/ rectangles 

\[ f = f(x) \]

Formulas:

Divide into \( n \) pieces.

Width of each is \( \Delta x = \frac{b-a}{n} \)

Right end pt. of \( i^{th} \) subinterval is \( a + i \Delta x \)

Left pt. \( i^{th} \) subinterval is \( a + (i-1) \Delta x \)

Midpt. \( i^{th} \) subinterval is \( \frac{1}{2} (a + i \Delta x + a + (i-1) \Delta x) \)

\[ = a + \frac{2i-1}{2} \Delta x \]

So

\[ R_n = \sum_{i=1}^{n} \Delta x f(a + i \Delta x) \]

\[ L_n = \sum_{i=1}^{n} \Delta x f(a + (i-1) \Delta x) \]

\[ M_n = \sum_{i=1}^{n} \Delta x f(a + \frac{2i-1}{2} \Delta x) \]

You did some ass. on yesterday's worksheet - more on WebAssign for tomorrow.
Riemann sum

\[ \Sigma_{i=1}^{n} f(x_i^*) \Delta x_i \]

\( x_i^* \) = any point in subinterval \( i \)

In our textbook, all subintervals have equal length

\( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \)

If limit is same for all choices of \( x_i^* \)

\[ \int_{a}^{b} f(x) \, dx \text{ means} \]

\( \text{This is the definition of a major topic in calculus.} \)

Probably important to learn!
Use right-hand Riemann sums to estimate the area under the curve \( f(x) = x^2 \) on the interval \([2, 8]\) with 3 rectangles.

\[
\Delta x = \frac{b-a}{n} = \frac{8-2}{3} = \frac{6}{3} = 2
\]

Base of each rectangle.

\[
R_3 = f(4) \cdot 2 + f(6) \cdot 2 + f(8) \cdot 2 = (4)^2 \cdot 2 + (6)^2 \cdot 2 + (8)^2 \cdot 2 = 232 \quad \text{(overestimate)}
\]
Basic Sums

\[ \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n-1) + n \]

\[ = \frac{n}{2} \cdot (n+1) = \frac{n(n+1)}{2} \]

\[ \sum_{i=1}^{n} i^2 = (1)^2 + (2)^2 + \ldots + (n-1)^2 + n^2 \]

\[ = \frac{n(n+1)(2n+1)}{6} \]

\[ \sum_{i=1}^{n} i^3 = (1)^3 + (2)^3 + (3)^3 + \ldots + n^3 \]

\[ = \left[ \frac{n(n+1)}{2} \right]^2 \]

Also:

\[ \sum_{i=1}^{n} i^{n-times} = \underbrace{y + y + \ldots + y}_{n-times} \]

\[ = 4(n) \]

\[ = 4n \]
Use formulas for basic sums to find the following:

\[ \sum_{k=1}^{200} 6k = 6 \cdot \sum_{k=1}^{200} k = 6 \left[ \frac{(200)(200+1)}{2} \right] = 120,600 \]

\[ \sum_{k=4}^{100} k^3 = \sum_{k=1}^{100} k^3 - \sum_{k=1}^{3} k^3 = \left[ \frac{100(100+1)^2}{2} \right] - \left[ \frac{3(3+1)^2}{2} \right] = 25502464 \]

\[ \sum_{k=1}^{n} 10k + 4 = 10 \sum_{k=1}^{n} k + 4n = 10 \left( \frac{n(n+1)}{2} \right) + 4n \]
Find the exact area under the curve \( f(x) = x^2 \) on the interval \([2, 8]\).

\[
\Delta x = \frac{b-a}{n} = \frac{8-2}{n} = \frac{6}{n}
\]

\[
R_n = \sum_{k=1}^{n} f\left(2 + k \cdot \frac{6}{n}\right) \cdot \frac{\Delta x}{n}
\]

\[
= \sum_{k=1}^{n} \left(2 + k \cdot \frac{6}{n}\right)^2 \cdot \frac{6}{n}
\]

\[
= \sum_{k=1}^{n} \left(4 + 2 \frac{6k}{n} + \frac{36k^2}{n^2}\right) \cdot \frac{6}{n}
\]

\[
= \sum_{k=1}^{n} \frac{4}{n} + \frac{144k}{n^2} + \frac{36k^2}{n^3}
\]

\[
= \sum_{k=1}^{n} \frac{144k}{n^2} + \frac{36k^2}{n^3}
\]

\[
= \frac{24}{n} \sum_{k=1}^{n} \frac{1}{n} + \frac{144}{n^2} \sum_{k=1}^{n} \frac{k}{n^2} + \frac{36}{n^3} \sum_{k=1}^{n} \frac{k^2}{n^3}
\]

\[
= \frac{24}{n} \cdot \frac{n}{2} + \frac{144}{n^2} \left(\frac{n(n+1)}{2}\right) + \frac{36}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}
\]
\[ R_n = 24 + 1 \left( \frac{72 n^2 + 72n}{n^2} + \frac{36n}{n^2} (n+1)(2n+1) \right) \]

\[ R_n = 24 + \frac{72 n^2 + 72n}{n^2} + \frac{36}{n^2} (2n^2 + 3n + 1) \]

\[ \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( 24 + \frac{72 + \frac{72}{n}}{n^2} \cdot \frac{n^2}{n^2} + \frac{72 + \frac{36 \cdot 3}{n} + \frac{36}{n^2}}{n^2} \right) \]

\[ = 24 + 72 + 72 = 168 \]
Calculate \( \int_{-3}^{3} (x^2 - 9) \, dx \)

What is the area between the curve and the x-axis for the function \( x^2 - 9 \) on the interval \([-3, 3]\)?

\[
\int_{-3}^{3} (x^2 - 9) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

we set up the sum as a right endpoint approx

\[
R_n = \sum_{i=1}^{n} f(x_i) \Delta x \quad \Delta x = \frac{b-a}{n} = \frac{3-(-3)}{n}
\]

\[= \frac{6}{n}\]

\[x_i = -3 + \frac{i}{n}\]

\[
= \sum_{i=1}^{n} f(-3 + \frac{i}{n}) \cdot \frac{6}{n}
\]

\[
= \sum_{i=1}^{n} \left[ (-3 + \frac{i}{n})^2 - 9 \right] \frac{6}{n}
\]
\[
\sum_{i=1}^{n} \left[ (q + a \left(3 \times \frac{6\cdot i}{n}\right) + \frac{3\cdot i^2}{n^2}) - q \right] \left( \frac{6}{n} \right)
\]

\[
\sum_{i=1}^{n} \frac{q\cdot 6}{n} \cdot \frac{6\cdot i}{n^2} + \frac{(36)(6)\cdot i^2}{n^3} - \frac{q\cdot 6}{n}
\]

\[-\frac{36(6)}{n^2} \sum_{i=1}^{n} i + \frac{(36)(6)}{n^3} \sum_{i=1}^{n} i^2
\]

\[-\frac{36(6)^3}{n^2} \cdot \frac{n\cdot (n+1)}{2} + \frac{(36)(6)}{n^3} \frac{n\cdot (n+1)\cdot (2n+1)}{6}
\]

\[
R_n = -108n^2 - 108 + 72n^3 + 36n^2 - 7n^2 + 36n
\]

\[
R_n = -36
\]
So the definite integral
\[
\int_{-3}^{3} (x^2 - 9) \, dx = -36
\]
The area between the curve \( f(x) \) is
\[
+36
\]
B/c of "negative heights" in the sum, numerically calculating a definite integral might not equal the area under a curve, but rather give the "signed area."
If \( f \) is integrable on \([a,b]\),

\[
\int_{a}^{b} f(x) \, dx
\]

exists.

\[
\text{Then, if } f \text{ is continuous on } [a,b],
\text{then } f \text{ is integrable on } [a,b].
\]

If \( f \) is integrable, can choose any \( x_i^* \), usually right endpoint, to compute.

We will need summation formulas (show page from book) and properties (other page).