Lecture 15

**Newton's Method** - 1600's
- A way to use derivatives (then a new invention) to find roots of equations
- This method and variations of it are still used in computer algorithms (i.e. in graphing calculators)

Example: “Find the roots of $x^5 - x + 3 = 0$”

No algebraic method

Idea of Newton's Method to find roots
- Start w/ approximation $x_1$ (f($x_1$) near 0)
- Find tangent line to $y = f(x)$ at $x_1$
- Find the x-intercept of the tangent line (this ends up being a better approx of our root, usually)
- Call this intercept $x_2$
- Repeat the process w/ $x_2$ to find $x_3$
Each new approximation is closer to the actual root.

Find a formula for $x_2$, given $x_1$:

- We make the eqn of the tangent line to $y = f(x)$ at $x_1$.
  
  $\text{slope: } f'(x_1) \quad \text{point } (x_1, f(x_1))$

  $y - f(x_1) = f'(x_1)(x - x_1)$

- Find $x$-int ($y = 0$) of the tangent line

  $0 - f(x_1) = f'(x_1)(x - x_1)$
  
  $-\frac{f(x_1)}{f'(x_1)} = x - x_1$
  
  $x = x_1 - \frac{f(x_1)}{f'(x_1)}$
Call the x-int $X_2$

$$X_2 = X_1 - \frac{f(x_1)}{f'(x_1)}$$

$$X_3 = X_2 - \frac{f(x_2)}{f'(x_2)}$$

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$$

What could go wrong:

- $f'(x_n) = 0$ brings process to a halt
- $f'(x_n)$ close to 0 may lead you way off

starting w/ $f'(x_1)$ near 0.
Approximate $\sqrt[6]{2}$ to 8 decimals

$x = \sqrt[6]{2}$
$x^6 = 2$
$x^6 - 2 = 0$

$\sqrt[6]{2}$ is a root of this eqn.

We use Newton's method to approximate this root.

$f'(x) = 6x^5$
Choose $x_1$, (I pick)

$x_1 = 1$

$x_2 = 1 - \frac{1 - 2}{6} = 1.166666667$

$x_3 = 1.12644368$

$x_4 = 1.12249707$

$x_5 = \{ 1.12246205 \text{ Same stop.} \}$

$x_6 = 1.12246205$
Anti-Derivatives

**Defn.** \( F(x) \) is an anti-derivative of \( f(x) \) (on an interval) if \( F'(x) = f(x) \)

**Ex.**  
- \( f(x) = x^3 \)  
  - \( F(x) = \frac{1}{4} x^4 + C \)

- \( f(x) = \sec^2(x) + \frac{1}{x^2} \)  
  - \( F(x) = \tan(x) - x^{-1} + C \)
### Table of Antiderivatives

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$F(x)$</th>
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</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1}$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sin(x)$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$-\cos(x)$</td>
</tr>
<tr>
<td>$\sec^2(x)$</td>
<td>$\tan(x)$</td>
</tr>
<tr>
<td>$\csc^2(x)$</td>
<td>$-\cot(x)$</td>
</tr>
<tr>
<td>$\sec(x)\tan(x)$</td>
<td>$\sec(x)$</td>
</tr>
<tr>
<td>$(\sec(x))(\cot(x))$</td>
<td>$-\csc(x)$</td>
</tr>
<tr>
<td>$\frac{1}{1+x^2}$</td>
<td>$\arctan(x)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\arcsin(x)$</td>
</tr>
</tbody>
</table>
An environmentalist finds that a certain type of tree grows in such a way that its height, \( h(t) \), after \( t \) years, is changing at a rate of

\[
h'(t) = 0.2t^{2/3} + \sqrt{t}
\]

If the tree was 2 feet tall when it was planted, how tall will it be in 27 years?

\[
h(t) = 0.2 \left( \frac{1}{\sqrt[3]{5}} \right) t^{5/3} + \frac{1}{3} t^{3/2} + C
\]

\[
= \frac{3}{25} t^{5/3} + \frac{2}{3} t^{3/2} + C
\]

must pass through \((0, 2)\)

\[\]

\[2 = 0 + 0 + C\]

\[2 = C\]

\[
h(t) = \frac{3}{25} t^{5/3} + \frac{2}{3} t^{3/2} + 2
\]

\[
h(27) \approx 124.7 \text{ feet}
\]
Rewrite algebraically, then find anti-derivatives

\[ g(x) = \frac{(x^3 + 6/x)}{x^3} \]

\[ g(x) = \frac{x^6 + a(6/x) + bx^3}{x^3} + \frac{3b/x^2}{x^3} \]

\[ = x^3 + 12x^2 + 3b/x^2 \]

\[ = x^3 + 12x^{-1} + 36x^{-3} \]

\[ G(x) = \frac{1}{3}x^4 + 12\ln|x| + \frac{1}{9}36x^{-4} + C \]

b) \[ h(\theta) = \frac{\cos(\theta)}{\sin^{-1}(\theta)} \]

\[ = \frac{1}{\sin(\theta)} \cdot \frac{\cos(\theta)}{\sin(\theta)} \]

\[ = (\csc(\theta) \cot(\theta)) \]

\[ H(\theta) = -\csc(\theta) + C \]
\[ f(x) = \frac{5x^2 + 8}{x^2 + 1} \]

\[ = \frac{5(x^2 + 1) + 3}{x^2 + 1} \]

\[ = \frac{5(x^2 + 1)}{x^2 + 1} + \frac{3}{x^2 + 1} \]

\[ = 5 + 3 \cdot \frac{1}{x^2 + 1} \]

\[ F(x) = 5x + 3 \arctan(x) + C \]

\[ \frac{5}{x^2 + 1} \left( \frac{5x^2 + 8}{x^2 + 1} \right) - (5x + 5) \]

\[ = 3 \cdot \frac{5x^2 + 8}{x^2 + 1} = 5 + \frac{3}{x^2 + 1} \]

\[ f(x) = 5 + \frac{3}{x^2 + 1} \]

\[ F(x) = 5x + 3 \cdot \arctan(x) + C \]
Ex: A stone is dropped from a cliff. It hits the ground with speed of 120 ft/sec. What is the height of the cliff?

**Acceleration due to gravity is constant**

\[ a(t) = -32 \text{ ft/sec}^2 \quad (-9.8 \text{ m/sec}^2) \]

\[ v(t) = -32t + C \]

At \( t=0 \), \( v=0 \)

\[ 0 = -32(0) + C \]

\[ C = 0 \]

\[ v(t) = -32t \]

\[ s(t) = -\frac{32}{2}t^2 + D \]

At \( t=0 \), we have the **height of the cliff**

\[ s(0) = 0 + D \]

When does the stone hit the ground?

\[ 0 = -16t^2 + D \]

\[ D = 16t^2 \]

\[ t^2 = \frac{D}{16} \]

\[ t = \pm \sqrt{\frac{D}{16}} \]

At this time, velocity = -120

\[ -120 = -32 \left( \frac{D}{4} \right) \]

\[ -120 = \sqrt{D} = 15 \]

\[ D = 225 \]

**Cliff is 225 ft high**