Lecture 14

A rectangle is to be inscribed in a semi-circle of radius 2. What is the largest possible area, and what dimensions give this area?

width = x
length = 2y

Optimization Eqn
Area = (2y)(x) = 2xy

Want to maximize area, but our function has 2 variables! We need a substitution

Constraint Eqn
\[ x^2 + y^2 = 2^2 \]
\[ y = + \sqrt{4-x^2} \]

\[ A(x) = 2x(\sqrt{4-x^2}) \]
\[ A'(x) = (2x)(\frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)) + \sqrt{4-x^2}(2) \]
\[ = -2x^2 \frac{1}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} \]

Domain
0 ≤ x ≤ 2
\[ A'(x) = 0 \quad \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0 \]

\[ \sqrt{4-x^2} = \frac{x^2}{\sqrt{4-x^2}} \]

\[ 4-x^2 = x^2 \]
\[ 4 = 2x^2 \]
\[ x^2 = 2 \]
\[ x = \pm \sqrt{2} \]

Only \( x = +\sqrt{2} \) is in the domain

\[
\begin{array}{c|c}
 x & A(x) = 2x\left(\sqrt{4-x^2}\right) \\
\hline
 0 & 0 \\
 2 & 0 \\
 \sqrt{2} & 2\sqrt{2}\left(\sqrt{4-2}\right) = 4
\end{array}
\]

Abs Max area of rectangle is 4

**Dimensions of rectangle of abs max area:**

- Width: \( \sqrt{2} \)
- Length: \( 2\sqrt{2} \)

[2\( \sqrt{2} \) by 2]
A right circular cylinder is inscribed in a cone with height \( h \) and base radius \( r \). Find the largest possible volume of such a cylinder.

\( (h, r \) are constants)\\
\[ \begin{align*}
\text{x} &= \text{radius of cylinder} \\
\text{y} &= \text{height of cylinder}
\end{align*} \]

**Optimization Eqn**

\[ \text{Vol} = \pi x^2 y \]

**By Similar Triangles,**

**Constraint Eqn**

\[ \frac{h}{r} = \frac{h-y}{x} \]

\[ \begin{align*}
xh &= r(h-y) \\
x &= \frac{r}{h}(h-y)
\end{align*} \]

\[ \begin{align*}
V &= \pi \left[ \frac{r}{h}(h-y) \right]^2 y \\
&= \pi \frac{r^2}{h^2} (h-y)^2 y = \pi \frac{r^2}{h^2} \left[ y^3 - 2hy^2 + h^2 y \right]
\end{align*} \]

**Domain**

\( 0 \leq y \leq h \)
\[
\frac{dV}{dy} = \frac{\pi r^2}{h^2} \left[ 3y^2 - 4hy + h^2 \right]
\]

\[
= \frac{\pi r^2}{h^2} \left[ 3y - h \right] \left[ y - h \right]
\]

Critical #s \[ \left( V'(y) = 0 \right) \]
\[
3y - h = 0 \quad y - h = 0
\]
\[
3y = h \quad y = h
\]
\[
y = \frac{h}{3}
\]

Want: Absolute max of \( V(y) \) when \( 0 \leq y \leq h \)

\[
y \quad V(y) = \pi \left[ \frac{r}{h} (h-y) \right]^2 y
\]

\[
\begin{array}{c|c}
0 & 0 \\
\frac{h}{3} & \pi \left( \frac{r}{h} \cdot \frac{2}{3} h \right)^2 \frac{h}{3} = \frac{4}{9} \pi r^2 h \\
h & 0
\end{array}
\]
Find the point on the line $y = 2x + 4$ that is closest to the origin.

**Constraint Eqn**: $y = 2x + 4$

**Distance from $(x,y)$ to $(0,0)$**

$$D = \sqrt{(x-0)^2 + (y-0)^2}$$

Our optimization eqn w/ 2 variables

**Domain**: $-\infty < x < \infty$

**Note**: The point $(x,y)$ which minimizes $D$ also minimizes $D^2$

$$D^2 = x^2 + (2x+4)^2$$

$$D^2' = 2x + 2(2x+4)(2)$$

$$= 2x + 8x + 16$$

$$= 10x + 16$$

**Critical Points**

$$D^2' = 0$$

$$10x + 16 = 0$$

$$x = \frac{-16}{10} = -\frac{8}{5}$$
**Method 1**

There is a single critical point at $x = -8/5$

\[
\lim_{x \to \infty} D^2 = +\infty \quad \lim_{x \to -\infty} D^2 = +\infty
\]

So $D^2$ has an abs min at $x = -8/5$

**Method 2**

There is a single critical point at $x = -8/5$

\[
\begin{array}{c}
\leftarrow \quad 0 \quad + \\
-8/5
\end{array}
\]

$(D^2)' = 10x + 16$

$(D^2)'' = 10$

The graph is convex everywhere (in particular at $x = -8/5$)

So $D^2$ has an abs min at $x = -8/5$

**Method 3**

There is a single critical point at $x = -8/5$

\[
(D^2)' = 10x + 16
\]

\[
(D^2)'' = 10
\]

The graph is convex everywhere (in particular at $x = -8/5$)

So $D^2$ has an abs min at $x = -8/5$

\[
x = -8/5 \quad y = 2\left(-\frac{8}{5}\right) + 4 = 4/5
\]

**Final Answer:** \[\left(-\frac{8}{5}, \frac{4}{5}\right)\]