A differential eqn is an equation that includes derivatives:

\[ y'' + 3y' + y = x^2 \]
\[ \left( \frac{dy}{dx} \right)^2 - y = 3 \]

*You "solve" a differential eqn by finding a fcn \( y \) which satisfies the relationship given by the eqn.*

\[ y'' + 3y' + y = x^2 \]

(What fcn \( y \) has the property that)

(The second derivative of \( y \)) + (3 times the first derivative) + (the fcn) is \( x^2 \)?
Ex. Find formulas for the fcn $y$, given the following differential eqns and initial values

a) $\frac{dy}{dx} = 3e^x$  \hspace{1cm} y(0) = 8

\[ y = 3e^x + C \]

$y(0) = 8 \Rightarrow 8 = 3e^0 + C$

$8 = 3 + C$

$C = 5$

\[ y = 3e^x + 5 \]

b) $\frac{dy}{dx} = 2x$  \hspace{1cm} y(0) = 20

\[ y = 4x^2 + C \]

$y(0) = 20$

$20 = 0 + C$

$C = 20$

\[ y = 4x^2 + 20 \]

c) $\frac{dy}{dx} = 8y$  \hspace{1cm} y(0) = 20

What fcn has derivative 8 times itself?

\[ (e^x)' = e^x = y \]

\[ (e^{8x})' = e^{8x} \cdot 8 = 8y \]

\[ (2e^{8x})' = 2e^{8x} \cdot 8 = 8y \]

\[ y = Ce^{8x} \]

\[ y(0) = 20 \Rightarrow 20 = Ce^{8(0)} \]

\[ C = 20 \]

\[ y = 20e^{8x} \]
Let's summarize this solution type as a theorem

**Thm.** If \( \frac{dy}{dx} = ky \), then \( y = Ce^{kt} \)

Differential Eqs can be used to help understand how to compare growth rates of population size.

**Pop A** 100 people, increasing by 5 people a year.

**Pop B** 100,000, increasing by 5,000 people a year.

Which pop is growing "faster"?
Let's find their % change in population.

**Pop A** \( \frac{dP}{dt} = \frac{5}{100} = .05 \) 5%

**Pop B** \( \frac{dP}{dt} = \frac{5,000}{100,000} = .05 \) 5%

Each population is currently changing by 5% a year.

\[
\frac{dP}{dt} = .05P
\]

**P(0) = 100**

They have the same differential equations, different initial conditions.
Assuming the populations continue with a constant relative growth rate of 5% a year, we can write eqns modeling population size.

\[ \text{Pop A: } y = 100,000 \cdot e^{0.05t} \]

\[ \text{Pop B: } y = 100,000 \cdot e^{0.05t} \]

What is the size of Pop A after 10 years?

\[ y = 100,000 \cdot e^{0.05(10)} \approx 164.87 \]

Approx 165 people

\[ \text{Ex} \]

If a population is currently 200, growing at a constant relative growth rate of 3%, find a formula for population size.

\[ \frac{dp}{dt} = 0.03P \]

\[ P = Ce^{0.03t} \]

\[ P(0) = 200 \]

\[ P = 200e^{0.03t} \]
Ex: How long does it take for 100mg of caffeine in the bloodstream to be reduced to 10% of that amount? The 1/2 life of caffeine is 4 hrs for most people.

\[ y = Ce^{kt} \]

So, \( y = 100e^{k(4)} \)

\[ \frac{1}{2} = e^{4k} \]

\[ \ln\left(\frac{1}{2}\right) = \ln\left(e^{4k}\right) \]

\[ \ln\left(\frac{1}{2}\right) = 4k \cdot \ln(e) \]

\[ k = \frac{\ln\left(\frac{1}{2}\right)}{4} \]

\[ y = 100e^{t \cdot \frac{\ln\left(\frac{1}{2}\right)}{4}} \]

10 = 100e^{\frac{\ln\left(\frac{1}{2}\right)}{4} \cdot t}

\[ \frac{1}{10} = e^{\frac{\ln\left(\frac{1}{2}\right)}{4} \cdot t} \]

\[ \ln\left(\frac{1}{10}\right) = \ln\left(e^{\frac{\ln\left(\frac{1}{2}\right)}{4} \cdot t}\right) \]

\[ \ln\left(\frac{1}{10}\right) = \frac{\ln\left(\frac{1}{2}\right)}{4} \cdot t \]

\[ t = \frac{4 \ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)} \approx 13.3 \text{ hours} \]

We assume the same % of caffeine is disappearing from the bloodstream over time.

\[ \]
Newton’s Law of Cooling

\[ T = \text{temperature of an object} \]
\[ (\text{changing as it cools/warms}) \]
\[ (\text{variable}) \]

\[ T_s = \text{temperature of surroundings} \]
\[ (\text{constant}) \]

The rate of change of \( T \) is proportional to the difference between \( T \) and \( T_s \)

Let’s write this info as a mathematical statement

\[ \frac{dT}{dt} = \text{rate of change of } T \]

\[ T - T_s = \text{difference between } T \text{ and } T_s \]

\[ \frac{dT}{dt} = k (T - T_s) \]

Things are proportional if they differ by a constant multiple
\[ \frac{dT}{dt} = k (T - T_s) \]

\[ \frac{d(T - T_s)}{dt} = k (T - T_s) \]

Because \( T_s \) is a constant, the rate of change of \( T \) is the rate of change of \( T - T_s \).

Our theorem from earlier gives the solution of this differential eqn.

\[ T - T_s = \frac{k}{C} \left( e^{kt} \right) \]

\[ C = \text{what happens at time } t=0 \]

At \( t=0 \), \( T - T_s = T(0) - T_s \)

\[ T = T_s + (T(0) - T_s) e^{kt} \]

\( T \) or Newton's Law of Cooling
Ex. (14, §2.8) In a murder investigation, the temperature of a corpse was 32.5°C at 1:30 pm, and 30.3°C an hour later. Normal body temp is 37°C, and the temperature of the surroundings was 20°C. When did the murder take place?

\[ T = T_s + (T(0) - T_s) e^{kt} \]

\[ T_s = 20 \]
\[ T(0) = 32.5 \text{ (at } t=0 \text{ of 1:30 pm)} \]

\[ T = 20 + 12.5 e^{kt} \]

Use \( T(1) = 30.3 \) to solve for \( k \)

\[ 30.3 = 20 + 12.5 e^{k(1)} \]
\[ \frac{10.3}{12.5} = e^k \]
\[ \ln \left( \frac{10.3}{12.5} \right) = \ln (e^k) \]
\[ k = \ln \left( \frac{10.3}{12.5} \right) \approx -0.1936 \]

\[ T = 20 + 12.5 e^{-0.1936t} \]

Find \( t \) when \( T = 37 \)

\[ 37 = 20 + 12.5 e^{-0.1936t} \]
\[ 17 = -0.1936t \]
\[ \frac{17}{-0.1936} = e \]
\[ \ln \left( \frac{17}{12.5} \right) = \ln \left( e^{-0.1936t} \right) \]

A little before 12:00 pm