Lecture 6

**Ex** Determine the equation of the line which is tangent to \( f(x) = x^4 + 6x + 10 \) and perpendicular to the line \( x + 2y = 16 \).

An eqn of a line \( y - y_0 = m \ (x - x_0) \)

Slope is \( \perp \) to \( x + 2y = 16 \)

\[
\begin{align*}
2y &= -x + 16 \\
y &= -\frac{1}{2}x + 8
\end{align*}
\]

Want \( m = \frac{1}{2} \) (opposite reciprocal)

The "point" is the spot on the curve where the tangent line has slope \( 2 \).

\( f(x) = x^4 + 6x + 10 \)

\( f'(x) = 4x^3 + 6 \)

\[
\begin{align*}
4x^3 + 6 &= 0 \\
x^3 &= -\frac{3}{2} \\
x &= -\frac{\sqrt{6}}{2}
\end{align*}
\]

Point: \( (x_0, y_0) = (-1, f(-1)) \)

\[
\begin{align*}
f(-1) &= (-1)^4 + 6(-1) + 10 = 5
\end{align*}
\]

\[
y - 5 = 2(x + 1)
\]
**Product Rule**

Let $f(x) = u(x) \cdot v(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h) \cdot v(x+h) - u(x) \cdot v(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h) \cdot v(x+h) - u(x+h) \cdot v(x) + u(x+h) \cdot v(x) - u(x) \cdot v(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h) \cdot [v(x+h) - v(x)] + [u(x+h) - u(x)] \cdot v(x)}{h}$$

$$= \lim_{h \to 0} \frac{[\lim_{h \to 0} u(x+h)] \cdot [\lim_{h \to 0} v(x+h) - v(x)] + [\lim_{h \to 0} u(x+h) - u(x)] \cdot v(x)}{h}$$

$$= u(x) \cdot v'(x) + u'(x) \cdot v(x)$$

$$(u(x) \cdot v(x))' = u(x) \cdot v'(x) + u'(x) \cdot v(x)$$
The proofs of the Quotient Rule and Chain Rule involve similar techniques.

**Quotient Rule**

\[ h(x) = \frac{f(x)}{g(x)} \quad h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \]

**Chain Rule**

\[ h(x) = f(g(x)) \quad h'(x) = f'(g(x)) \cdot g'(x) \]

We can use ideas about inverse functions and the chain rule to find the derivatives of \( \ln(x) \) and \( \arcsin(x) \).

**Remember:**

\[ f(f^{-1}(x)) = x \]

\[ f^{-1}(f(x)) = x \]
Finding the derivative of $\ln(x)$

$e^x$ and $\ln(x)$ are inverse functions

\[
X = x \\
e^{\ln(x)} = x \quad \text{(Tautology using "inverse functions")}
\]

\[
\frac{d}{dx} (e^{\ln(x)}) = \frac{d}{dx} (x)
\]

use the chain rule

\[
e^{\ln(x)} \cdot (\ln(x))' = 1
\]

\[
(\ln(x))' = \frac{1}{e^{\ln(x)}}
\]

\[
= \frac{1}{x}
\]

\[
\frac{d}{dx} [\ln(x)] = \frac{1}{x}
\]

\[
\log_b(x) = \frac{\ln(x)}{\ln(b)} \quad \frac{d}{dx} \left[ \log_b(x) \right] = \frac{d}{dx} \left[ \frac{1}{\ln(b)} \cdot \ln(x) \right]
\]

\[
= \frac{1}{\ln(b)} \cdot \frac{1}{x}
\]

\[
= \frac{1}{x \ln(b)}
\]
Finding the derivative of $\arcsin(x)$

\[ x = x \]

\[ \sin(\arcsin(x)) = x \]

\[ \frac{d}{dx} \left( \sin(\arcsin(x)) \right) = \frac{d}{dx} (x) \]

\[ \cos(\arcsin(x)) \cdot (\arcsin(x))' = 1 \]

\[ (\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} \]

\[ \Theta = \arcsin(x) \]

\[ \sin \Theta = x = \frac{opp}{hyp} \]

\[ 1 \]

\[ \begin{array}{c}
\Theta \\
\sqrt{1-x^2} \\
x \\
\text{Pythagorean Thm}
\end{array} \]

\[ a^2 + x^2 = 1 \]
\[ a^2 = 1 - x^2 \]
\[ a = \pm \sqrt{1-x^2} \]

\[ \cos(\arcsin(x)) = \cos(\Theta) = \sqrt{1-x^2} \]

\[ \left[ \arcsin(x) \right]' = \frac{1}{\sqrt{1-x^2}} \]
Now, we will begin to show that
\[ \frac{d}{dx} (\sin(x)) = \cos(x) \]

To do this, we will need the identity
\[ \sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \]

In the process, we need to prove the following "special limits"
\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0 \]

As \( \theta \to 0 \), both fractions approach \( \frac{0}{0} \), and indeterminant form
Let \( f(x) = \sin(x) \). Find \( f'(x) \)

\[
f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}
\]

= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}

= \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h}

= \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}

= \lim_{h \to 0} \sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h}

= \left[ \lim_{h \to 0} \sin(x) \right] \left[ \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right] + \left[ \lim_{h \to 0} \cos(x) \right] \left[ \lim_{h \to 0} \frac{\sin(h)}{h} \right]

= \sin(x) \cdot 0 + \cos(x) \cdot 1

We must now show \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \) and \( \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0 \).
To show \( \lim_{{h \to 0}} \frac{\sin(h)}{h} = 1 \), we will use facts about:
- right triangles
- sectors of circles
- the squeeze theorem

Consider a sector of a circle of angle \( \varphi \):

\[ C = 2\pi r, \quad \widehat{AB} = \frac{\theta}{2\pi} (2\pi r) = \varphi r \]

A proportional fraction of the circumference.

Note:

Arc length \( \widehat{AB} = h \)

Using \( \triangle OBC \),

\[ \sin(h) = \frac{BC}{h} \]

Notice \( BC < \widehat{AB} = h \)

We now have an upper bound for the function.
Now create a larger triangle outside the sector

Draw a ⊥ line from point B to \( \overline{AD} \)

Call the point of intersection \( E \)

\[ h = \overline{AB} < \overline{AE} + \overline{EB} \]

\[ < \overline{AE} + \overline{ED} \]

\[ = \overline{AD} \]

\[ = \tan(h) \quad \tan(h) = \frac{\overline{AD}}{1} \]

\[ h < \tan(h) \]

\[ h < \sin(h) \]

\[ \frac{\cos(h)}{h} \]

\[ \cos(h) < \sin(h) \]

Now we have a lower bound for the function
\[\text{Want } \lim_{h \to 0} \frac{\sin(h)}{h}\]

\[
\cos(h) < \frac{\sin(h)}{h} < 1
\]

\[
\lim_{h \to 0} \cos(h) = 1
\]

\[
\lim_{h \to 0} 1 = 1
\]

So by the Squeeze Theorem

\[
\lim_{h \to 0} \frac{\sin(h)}{h} = 1
\]

Tuesday we will show \[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0
\]

(This is a short, algebraic technique)