Lecture 6

Example: Determine the equation of the line which is tangent to \( f(x) = x^4 + 6x + 10 \) and perpendicular to the line \( x + 2y = 16 \).

An eqn of a line: \( y - y_o = m (x - x_o) \)

Slope is \( \perp \) to \( x + 2y = 16 \)
\[ 2y = -x + 16 \]
\[ y = -\frac{1}{2}x + 8 \]

Want Slope \( m = 2 \) (Opposite reciprocal)

The "point" is the spot on the curve where the tangent line has slope 2.

\( f(x) = x^4 + 6x + 10 \)
\( f'(x) = 4x^3 + 6 \)

\[ 4x^3 + 6 = 2 \]
\[ 4x^3 = -4 \]
\[ x^3 = -1 \]
\[ x = -1 \]

Point: \( (x_0, y_0) = (-1, f(-1)) \)
\[ f(-1) = (-1)^4 + 6(-1) + 10 = 5 \]

\[ y - 5 = 2(x + 1) \]
**Chain Rule**

Describes how to take the derivative of a "composite func" aka fn "plugged into" another fn

If \( h(x) = f(g(x)) \)

Then \( h'(x) = f'(g(x)) \cdot g'(x) \)

**Ex** \( f(x) = (x^2 - \sqrt{x} + \tan(x))^{42} \)

\( y = x^{42} \)
\( y' = 42x^{41} \)

\[ f'(x) = 42(x^2 - x^{1/2} + \tan(x))^{41} \cdot (7x^6 - \frac{1}{2}x^{-1/2} + \sec^2(x)) \]

**Ex** \( f(x) = e^{\sin^2(x)} \)

\( y = e^x \)
\( y' = e^x \)

\[ f'(x) = e^{\sin^2(x)} \cdot \frac{\sin(2x)}{\sin(x)} \cdot (\sin^2(x))' \]

This derivative also involves the chain rule!

\[ = e^{\sin^2(x)} \cdot (2 \sin(x) \cdot \cos(x)) \]
We can use ideas about inverse functions and the chain rule to find the derivatives of $\ln(x)$ or $\arcsin(x)$.

Remember:

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$
Finding the derivative of $\ln(x)$

$e^x$ and $\ln(x)$ are inverse functions.

$x = x$

$e^{\ln(x)} = x$ (Tautology using "inverse funs")

$\frac{d}{dx} (e^{\ln(x)}) = \frac{d}{dx} (x)$

Use the chain rule

$e^{\ln(x)} \cdot (\ln(x))' = 1$

$(\ln(x))' = \frac{1}{e^{\ln(x)}}$

$= \frac{1}{x}$

$\frac{d}{dx} [\ln(x)] = \frac{1}{x}$

$log_b(x) = \frac{\ln(x)}{\ln(b)}$

$\frac{d}{dx} [\log_b(x)] = \frac{d}{dx} \left[ \frac{1}{\ln(b)} \cdot \ln(x) \right]$

$= \frac{1}{\ln(b)} \cdot \frac{1}{x}$

$= \frac{1}{x \ln(b)}$

These details not shown in class.
Finding the derivative of $\arcsin(x)$

$$x = x$$

$$\sin(\arcsin(x)) = x$$

$$\frac{d}{dx} \left( \sin(\arcsin(x)) \right) = \frac{d}{dx} (x)$$

$$\cos(\arcsin(x)) \cdot (\arcsin(x))' = 1$$

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$$

We use right triangles to figure this out.

$$\Theta = \arcsin(x)$$

$$\sin \Theta = \frac{\text{opp}}{\text{hyp}} = x$$

We use the Pythagorean Theorem:

$$a^2 + x^2 = 1$$

$$a^2 = 1 - x^2$$

$$a = \pm \sqrt{1-x^2}$$

$$\cos(\arcsin(x)) = \cos(\Theta) = \sqrt{1-x^2}$$

$$[\arcsin(x)]' = \frac{1}{\sqrt{1-x^2}}$$
Now, we will begin to show that

$$\frac{d}{dx} (\sin(x)) = \cos(x)$$

To do this, we will need the identity

$$\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

In the process, we need to prove the following "special limits":

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

As $\theta \to 0$, both fractions approach $\frac{0}{0}$, and indeterminate form.
Let $f(x) = \sin(x)$. Find $f''(x)$

\[
f''(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin(x) \cos(h) - \sin(x) + \cos(x) \sin(h)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin(x) [\cos(h) - 1] + \cos(x) \sin(h)}{h}
\]

\[
= \lim_{h \to 0} \sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h}
\]

\[
= \left[ \lim_{h \to 0} \sin(x) \right] \left[ \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right] + \left[ \lim_{h \to 0} \cos(x) \right] \left[ \lim_{h \to 0} \frac{\sin(h)}{h} \right]
\]

\[
= \sin(x) \cdot 0 + \cos(x) - 1
\]

We must now show $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$

\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0
\]
The shortest proof of the special limits involves L'Hopital's Rule.

\[ \text{L'Hopital's Rule} \rightarrow \text{coming later in the semester} \]

If \( \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \).

- \( \lim_{h \to 0} \frac{\sin(h)}{h} = \lim_{h \to 0} \frac{\cos(h)}{1} = \cos(0) = 1 \)
- \( \lim_{h \to 0} \frac{\cos(h)-1}{h} = \lim_{h \to 0} \frac{-\sin(h)}{1} = -\sin(0) = 0 \)

**But Wait!** In this proof of the limit, we use that \( (\sin(x))' = \cos(x) \).
But we need the limit to show \( (\sin(x))' = \cos(x) \). We need to prove this limit using another method, or we have circular logic.
To show \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \), we will use facts about

- right triangles
- sectors of circles
- the squeeze theorem

Consider a sector of a circle of angle \( c \)

\[ C = 2\pi r \quad \widehat{AB} = \frac{\Theta}{2\pi} (2\pi r) \]

= \Theta \cdot r

a proportional fraction of the circumference

Note

arc length \( \widehat{AB} = h \)

Using \( \triangle OBC \),

\[ \sin(h) = \frac{BC}{\overline{BC}} \]

Notice \( \overline{BC} < \widehat{AB} = h \)

\[ \sin(h) < h \]

\[ \frac{\sin(h)}{h} < 1 \]

We now have an upper bound for the function
Now create a larger triangle outside the sector

Draw a \perp line from point B to \overline{AD}

Call the point of intersection E

\textbf{Notice} \ h = \overline{AB} \leq \overline{AE} + \overline{EB}

\leq \overline{AE} + \overline{ED}

= \overline{AD}

\div \tan(h) \quad \tan(h) = \frac{\overline{AD}}{1}

\ \ h < \tan(h)

h < \frac{\sin(h)}{\cos(h)}

\frac{\cos(h)}{\cos(h)} < \frac{\sin(h)}{h}

\textbf{Now we have a lower bound for the function}
Want \( \lim_{h \to 0} \frac{\sin(h)}{h} \)

\[
\cos(h) \leq \frac{\sin(h)}{h} \leq 1
\]

\[
\lim_{h \to 0} \cos(h) = 1
\]

\[
\cos(h) \leq \frac{\sin(h)}{h} \leq 1
\]

So by the Squeeze Theorem

\[
\lim_{h \to 0} \frac{\sin(h)}{h} = 1
\]

Tuesday we will show \( \lim_{h \to 0} \frac{\cos(h)-1}{h} = 0 \)

(This is a short, algebraic technique)