Lecture 85

* Check class webpage for "Derivatives Handout"

- Memorize all rules on handout immediately, though we will slowly prove/explain them over the next 2 weeks.

- We will use all the different notations listed.

Review

The slope of the tangent line to \( y = f(x) \) at the point \( x = a \)

\[
f'(a) = m_{\text{tan}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

The derivative of \( y = f(x) \) at \( x = a \)

\[
= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

A formula for the slope of the tangent line at an arbitrary point "x"

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
Let's look at the graphs of some functions and their derivative (which we previously calculated).

At $x=0$,

- $y = x^3$ with $y' = 3x^2$
- $y = |x|$ with $y' = 1$ at $x=0$
- $y = x$ and $y = -x$

The $y$-value outputted is the slope of the tangent line of the previous graph.

It is not connected to the slope of the tangent line in this graph.

At $x=0$:

- Continuous
- Not Differentiable

\{No output in derivative fcn\}
At \( x=0 \), Not Continuous, Not Differentiable

\[
y = \frac{1}{x}
\]

\[
y' = -\frac{1}{x^2}
\]

\( m = -1 \)

\( m = \text{positive} \)

The \( y \)-value outputted is the slope of the tangent line of the previous graph.

It is not connected w/ the slope of the tangent line of this graph.

Is there a graph that is
Not Continuous at \( x=a \)?
Differentiable

Turns out, this is not possible. How could you prove no such graph exists?
**Defn:** $f$ is **differentiable at** $x=a$ if $f'(a)$ exists.

- $f$ is **differentiable on an interval** if $f$ is differentiable at every point of the interval.

In our previous example, $f(x) = 1/x$ is **not differentiable** at $x=0$.

$f(x) = 1/x$ is **differentiable** on $(-\infty,0)$ and $(0,\infty)$.
**Thm** If \( f(x) \) is differentiable at \( x = a \)

Then \( f(x) \) is continuous at \( x = a \)

**Proof Ideas** Notice that differentiable and continuous both mean certain limits exist

\[
f(x) \text{ is differentiable at } x = a \text{ means } \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ this limit exists}
\]

\[
f(x) \text{ is continuous at } x = a \text{ means } \\
\lim_{x \to a} f(x) = f(a)
\]

i.e

\[
\lim_{x \to a} [f(x) - f(a)] = 0
\]

Use algebra to show that if the first limit exists, the second limit is 0.
We can use the definition of derivative to find patterns in the derivatives of different types of fcn's.

1. \( f(x) = c \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0
\]

\( f'(x) = 0 \)

2. \( f(x) = x \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1
\]

\( f'(x) = 1 \)

Not shown in class.
Review the expansion of \((a+b)^n\) using Pascal’s Triangle below.

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \]

\[ = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \ldots + x^n}{h} \]

all terms have a factor of \(h\) or higher

\[ = \lim_{h \to 0} \frac{nx^{n-1}h + \ldots}{h} \]

all terms have \(h^2\) or higher

\[ = \lim_{h \to 0} \frac{h(nx^{n-1} + \ldots)}{h} \]

all terms have a factor of "\(h^n\)" or higher

\[ = nx^{n-1} \]

\[ f'(x) = nx^{n-1} \]

\[
\begin{array}{cccc}
1 & 1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

\[
\begin{align*}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{align*}
\]

\[ f(x^2) = x^n, \quad f'(x) = nx^{n-1} \]

holds for any \(x\) (though this proof is only for \(n = \text{positive integer} \))
(4) \( f(x) = g(x) + h(x) \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[g(x+h)+h(x+h)] - [g(x)+h(x)]}{h}
\]

\[
= \lim_{h \to 0} \frac{g(x+h)-g(x)+h(x+h)-h(x)}{h} \]

\[
= \lim_{h \to 0} \left[ \frac{g(x+h)-g(x)}{h} + \frac{h(x+h)-h(x)}{h} \right] \]

\[
= \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h \to 0} \frac{h(x+h)-h(x)}{h} = g'(x) + h'(x)
\]

\[
f'(x) = g'(x) + h'(x)
\]

---

**EX** Calculate \( \frac{d}{dx}(xe^x + e^x) \)

\[
= \frac{d}{dx}(xe^x) + \frac{d}{dx}(e^x)
\]

\[
= e \cdot xe^{-1} + e^x
\]
**Product Rule**

Let \( f(x) = u(x) \cdot v(x) \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}.
\]

\[
= \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{u(x+h)[v(x+h) - v(x)]}{h} + \frac{u(x+h) - u(x)}{h} \cdot v(x)
\]

\[
= \lim_{h \to 0} \frac{u(x+h)}{h} \cdot \frac{v(x+h) - v(x)}{h} + \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \cdot v(x)
\]

\[
= u(x) \cdot v'(x) + u'(x) \cdot v(x)
\]

\[
(u(x) \cdot v(x))' = u(x) \cdot v'(x) + u'(x) \cdot v(x)
\]