Applications of IVT

- Estimating points of intersection for 2 graphs
- Estimating roots (x-intercepts) of functions

Ex: Show that there is a root of the equation \(4x^3 - 6x^2 + 3x - 2 = 0\) between \(x = 1\) and \(x = 2\).

We are looking for a number \(c\) such that \(f(c) = 0\).

In IVT, consider the interval \(a = 1, b = 2, N = 0\).

\[
\begin{align*}
f(1) &= 4 - 6 + 3 - 2 = -1 < 0 \\ f(2) &= 32 - 24 + 6 - 2 = 12 > 0
\end{align*}
\]

Since \(f(x)\) is a continuous polynomial, the IVT says \(\exists \ a \neq c\) between \(1 \leq 2\) \(1 < c < 2\) so that \(f(c) = 0\).

Using Guess & Check and a calculator, we can find this root more precisely:

\[
\begin{align*}
f(1.22) &= -0.007008 \\ f(1.23) &= 0.056008
\end{align*}
\]

So a root lies in the interval (1.22, 1.23).
IVT plays a role in how graphing devices work!

A computer calculates a finite # of points on a graph, and turns on the pixels that contain these calculated \((x,y)\) points.

Then it connects the dots! 

\(\text{Assumes the fcn is continuous}\)

\(\text{It uses IVT}\)

\[\star \] To estimate a point of intersection

\[f(x) = g(x)\]

\[\star \] Estimate roots of 

\[f(x)-g(x) = 0\]
Now that we have more experience w/ limits, let's revisit the tangent line problem.

Think of "a" as fixed and "x" as variable.

Slope of secant line $\overrightarrow{PQ} \quad M_{\overrightarrow{PQ}} = \frac{f(x) - f(a)}{x-a}$

As $x \to a$, $\overrightarrow{PQ}$ approaches the tangent line at $P(a, f(a))$

**Definition:** Slope of tangent line to $y = f(x)$ at $(a, f(a))$ is:

$$m_{\text{tan}} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

**New notation:**

$$f'(a) = m_{\text{tan}}$$

We call this the "derivative"
Find the derivative of $f(x) = \frac{1}{x}$ at $x=2$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{2x - x}{2x(x - 2)}}{x - 2}$$

$$= \lim_{x \to 2} \frac{2 - x}{2x(x - 2)}$$

$$= \lim_{x \to 2} \frac{(2-x)}{2x(x-2)}$$

$$= \lim_{x \to 2} \frac{- (x-2)}{2x(x-2)}$$

$$= \lim_{x \to 2} \frac{-1}{2x}$$

$$= -\frac{1}{4}$$

The slope is $-\frac{1}{4}$. 

$y = \frac{1}{x}$
Another Notation

\[ m_{PQ} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \]

As \( x \to a, \ h \to 0 \)

[Defn] Slope of tangent line to \( y = f(x) \) at \( (a, f(a)) \) is:

\[ M_{\text{tan}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

\[ f'(a) = M_{\text{tan}} \]
For a \( a \), \( \, f'(a) \) is a number.

Can we find a formula for \( f'(x) \) for any \( x \)?

**Defn.** \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

Use variable \( x \) instead of constant \( a \).

**Notation** \( \frac{dy}{dx} \)

**Ex.** \( f(x) = x^3 \). Find \( f'(x) \).

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2 + 0 + 0 \\
\end{align*}
\]

\( f'(x) = 3x^2 \)

\( f'(2) = 3(2)^2 = 12 \) \( \rightarrow \) slope of the tangent line to \( f(x) = x^3 \) when \( x = 2 \)
EX: \( f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \)

Find \( f'(x) \)

\( \text{If } x > 0, \quad \frac{f(x+h)-f(x)}{h} \rightarrow \frac{1}{0} \quad \text{(we can choose } h \text{ small enough so } x+h>0 \text{ for small } h) \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{|x+h|-|x|}{h}
\]

\[
= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = \boxed{1}
\]

\( \text{If } x < 0, \quad \frac{f(x+h)-f(x)}{h} \rightarrow \frac{-1}{0} \quad \text{(we can choose } h \text{ small enough so } x+h < 0 \text{ for small } h) \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{|x+h|-|x|}{h}
\]

\[
= \lim_{h \to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = \boxed{-1}
\]
What about $x=0$?

\[ f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \]

Right-hand limit
\[ \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \]

So \( \lim_{h \to 0^+} \frac{|h|}{h} = \text{DNE} \)

Left-hand limit
\[ \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0} -1 = -1 \]

for \( f(x) = |x| \)

\[ f'(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 \\
\text{DNE} & \text{if } x = 0
\end{cases} \]

Geometrically, we see there is no unique tangent line of \( f(x) = |x| \) at \( x = 0 \).
**Defn:** \( f \) is differentiable at \( x=a \) if
\( f'(a) \) exists

- \( f \) is differentiable on an interval if
  \( f \) is differentiable at every point of the interval

In our previous example,
\( f(x) = 1x \) is not differentiable at \( x=0 \)
is differentiable on \((-\infty, 0)\) and \((0, \infty)\)